***Remark*** A similar proof may be given for the continuity of cosine function.

**Example 18** Prove that the function defined by $f(x) = \tan x$ is a continuous function.

**Solution** The function $f(x) = \tan x = \dfrac{\sin x}{\cos x}$. This is defined for all real numbers such that $\cos x \neq 0$, i.e., $x \neq (2n+1)\dfrac{\pi}{2}$. We have just proved that both sine and cosine functions are continuous. Thus $\tan x$ being a quotient of two continuous functions is continuous wherever it is defined.

An interesting fact is the behaviour of continuous functions with respect to composition of functions. Recall that if $f$ and $g$ are two real functions, then

$$(f \circ g)(x) = f(g(x))$$

is defined whenever the range of $g$ is a subset of domain of $f$. The following theorem (stated without proof) captures the continuity of composite functions.

**Theorem 2** Suppose $f$ and $g$ are real valued functions such that $(f \circ g)$ is defined at $c$. If $g$ is continuous at $c$ and if $f$ is continuous at $g(c)$, then $(f \circ g)$ is continuous at $c$.

The following examples illustrate this theorem.

**Example 19** Show that the function defined by $f(x) = \sin(x^2)$ is a continuous function.

**Solution** Observe that the function is defined for every real number. The function $f$ may be thought of as a composition $g \circ h$ of the two functions $g$ and $h$, where $g(x) = \sin x$ and $h(x) = x^2$. Since both $g$ and $h$ are continuous functions, by Theorem 2, it can be deduced that $f$ is a continuous function.

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**Example 20** Show that the function $f$ defined by

$$f(x) = |1 - x + |x||,$$

where $x$ is any real number, is a continuous function.

**Solution** Define $g$ by $g(x) = 1 - x + |x|$ and $h$ by $h(x) = |x|$ for all real $x$. Then

$$\begin{aligned}(h \circ g)(x) &= h(g(x))\\ &= h(1 - x + |x|)\\ &= |1 - x + |x|| = f(x)\end{aligned}$$

In Example 7, we have seen that $h$ is a continuous function. Hence $g$ being a sum of a polynomial function and the modulus function is continuous. But then $f$ being a composite of two continuous functions is continuous.

or $$\frac{dy}{dx} = y \log a$$

Thus $$\frac{d}{dx}(a^x) = a^x \log a$$

**Alternatively** $$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\log a}) = e^{x\log a}\frac{d}{dx}(x\log a)$$

$$= e^{x \log a} \,.\, \log a = a^x \log a.$$

**Example 32** Differentiate $x^{\sin x}$, $x > 0$ w.r.t. $x$.

**Solution** Let $y = x^{\sin x}$. Taking logarithm on both sides, we have

$$\log y = \sin x \log x$$

Therefore $$\frac{1}{y}\cdot\frac{dy}{dx} = \sin x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(\sin x)$$

or $$\frac{1}{y}\frac{dy}{dx} = (\sin x)\frac{1}{x} + \log x \cos x$$

or $$\frac{dy}{dx} = y\left[\frac{\sin x}{x} + \cos x \log x\right]$$

$$= x^{\sin x}\left[\frac{\sin x}{x} + \cos x \log x\right]$$

$$= x^{\sin x - 1}\cdot\sin x + x^{\sin x}\cdot\cos x \log x$$

**Example 33** Find $\frac{dy}{dx}$, if $y^x + x^y + x^x = a^b$.

**Solution** Given that $y^x + x^y + x^x = a^b$.

Putting $u = y^x$, $v = x^y$ and $w = x^x$, we get $u + v + w = a^b$

Therefore $$\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0 \qquad \text{... (1)}$$

Now, $u = y^x$. Taking logarithm on both sides, we have

$$\log u = x \log y$$

Differentiating both sides w.r.t. $x$, we have

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$$\frac{1}{u}\cdot\frac{du}{dx} = x\frac{d}{dx}(\log y)+\log y\frac{d}{dx}(x)$$

$$= x\frac{1}{y}\cdot\frac{dy}{dx}+\log y\cdot 1$$

So
$$\frac{du}{dx} = u\left(\frac{x}{y}\frac{dy}{dx}+\log y\right) = y^x\left[\frac{x}{y}\frac{dy}{dx}+\log y\right] \quad \dots (2)$$

Also $v = x^y$

Taking logarithm on both sides, we have

$$\log v = y \log x$$

Differentiating both sides w.r.t. $x$, we have

$$\frac{1}{v}\cdot\frac{dv}{dx} = y\frac{d}{dx}(\log x)+\log x\frac{dy}{dx}$$

$$= y\cdot\frac{1}{x}+\log x\cdot\frac{dy}{dx}$$

So
$$\frac{dv}{dx} = v\left[\frac{y}{x}+\log x\frac{dy}{dx}\right]$$

$$= x^y\left[\frac{y}{x}+\log x\frac{dy}{dx}\right] \quad \dots (3)$$

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Again
$$w = x^x$$

Taking logarithm on both sides, we have

$$\log w = x \log x.$$

Differentiating both sides w.r.t. $x$, we have

$$\frac{1}{w}\cdot\frac{dw}{dx} = x\frac{d}{dx}(\log x)+\log x\cdot\frac{d}{dx}(x)$$

$$= x\cdot\frac{1}{x}+\log x\cdot 1$$

i.e.
$$\frac{dw}{dx} = w\,(1+\log x)$$

$$= x^x\,(1+\log x) \quad \dots (4)$$

From (1), (2), (3), (4), we have

$$y^x\left(\frac{x}{y}\frac{dy}{dx}+\log y\right)+x^y\left(\frac{y}{x}+\log x\frac{dy}{dx}\right)+x^x\,(1+\log x)=0$$

or $$(x\,.\,y^{x-1}+x^y\,.\,\log x)\,\frac{dy}{dx}=-\,x^x\,(1+\log x)-y\,.\,x^{y-1}-y^x\log y$$

Therefore $$\frac{dy}{dx}=\frac{-[y^x\log y+y\,.\,x^{y-1}+x^x\,(1+\log x)]}{x\,.\,y^{x-1}+x^y\log x}$$

## EXERCISE 5.5

Differentiate the functions given in Exercises 1 to 11 w.r.t. $x$.

**1.** $\cos x\,.\,\cos 2x\,.\,\cos 3x$

**2.** $\sqrt{\dfrac{(x-1)\,(x-2)}{(x-3)\,(x-4)\,(x-5)}}$

**3.** $(\log x)^{\cos x}$

**4.** $x^x - 2^{\sin x}$

**5.** $(x+3)^2\,.\,(x+4)^3\,.\,(x+5)^4$

**6.** $\left(x+\dfrac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

**7.** $(\log x)^x + x^{\log x}$

**8.** $(\sin x)^x + \sin^{-1}\sqrt{x}$

**9.** $x^{\sin x} + (\sin x)^{\cos x}$

**10.** $x^{x\cos x} + \dfrac{x^2+1}{x^2-1}$

**11.** $(x\cos x)^x + (x\sin x)^{\frac{1}{x}}$

Find $\dfrac{dy}{dx}$ of the functions given in Exercises 12 to 15.

**12.** $x^y + y^x = 1$

**13.** $y^x = x^y$

**14.** $(\cos x)^y = (\cos y)^x$

**15.** $xy = e^{(x-y)}$

**16.** Find the derivative of the function given by $f(x) = (1+x)\,(1+x^2)\,(1+x^4)\,(1+x^8)$ and hence find $f'(1)$.

**17.** Differentiate $(x^2 - 5x + 8)\,(x^3 + 7x + 9)$ in three ways mentioned below:

(i) by using product rule

(ii) by expanding the product to obtain a single polynomial.

(iii) by logarithmic differentiation.

Do they all give the same answer?

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Therefore $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta = -\sqrt[3]{\frac{y}{x}}$$

**Note** Had we proceeded in implicit way, it would have been quite tedious.

## EXERCISE 5.6

If $x$ and $y$ are connected parametrically by the equations given in Exercises 1 to 10, without eliminating the parameter, Find $\frac{dy}{dx}$.

**1.** $x = 2at^2,\ y = at^4$

**2.** $x = a\cos\theta,\ y = b\cos\theta$

**3.** $x = \sin t,\ y = \cos 2t$

**4.** $x = 4t,\ y = \frac{4}{t}$

**5.** $x = \cos\theta - \cos 2\theta,\ y = \sin\theta - \sin 2\theta$

**6.** $x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta)$

**7.** $x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$

**8.** $x = a\left(\cos t + \log\tan\frac{t}{2}\right),\ y = a\sin t$

**9.** $x = a\sec\theta,\ y = b\tan\theta$

**10.** $x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$

**11.** If $x = \sqrt{a^{\sin^{-1}t}},\ y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

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## 5.7 Second Order Derivative

Let $y = f(x)$. Then

$$\frac{dy}{dx} = f'(x) \qquad \text{... (1)}$$

If $f'(x)$ is differentiable, we may differentiate (1) again w.r.t. $x$. Then, the left hand side becomes $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ which is called the *second order derivative* of $y$ w.r.t. $x$ and is denoted by $\frac{d^2y}{dx^2}$. The second order derivative of $f(x)$ is denoted by $f''(x)$. It is also we need to find all $x$ such that $\dfrac{2^{x+1}}{1+4^x} \leq 1$, i.e., all $x$ such that $2^{x+1} \leq 1 + 4^x$. We may rewrite this as $2 \leq \dfrac{1}{2^x} + 2^x$ which is true for all $x$. Hence the function is defined at every real number. By putting $2^x = \tan\theta$, this function may be rewritten as

$$f(x) = \sin^{-1}\left[\frac{2^{x+1}}{1+4^x}\right]$$

$$= \sin^{-1}\left[\frac{2^x \cdot 2}{1+(2^x)^2}\right]$$

$$= \sin^{-1}\left[\frac{2\tan\theta}{1+\tan^2\theta}\right]$$

$$= \sin^{-1}[\sin 2\theta]$$

$$= 2\theta = 2\tan^{-1}(2^x)$$

Thus

$$f'(x) = 2 \cdot \frac{1}{1+(2^x)^2} \cdot \frac{d}{dx}(2^x)$$

$$= \frac{2}{1+4^x} \cdot (2^x)\log 2$$

$$= \frac{2^{x+1}\log 2}{1+4^x}$$

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**Example 46** Find $f'(x)$ if $f(x) = (\sin x)^{\sin x}$ for all $0 < x < \pi$.

**Solution** The function $y = (\sin x)^{\sin x}$ is defined for all positive real numbers. Taking logarithms, we have

$$\log y = \log(\sin x)^{\sin x} = \sin x \log(\sin x)$$

Then

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\sin x \log(\sin x))$$

$$= \cos x \log(\sin x) + \sin x \,.\, \frac{1}{\sin x} \cdot \frac{d}{dx}(\sin x)$$

$$= \cos x \log(\sin x) + \cos x$$

$$= (1 + \log(\sin x)) \cos x$$

- Chain rule is rule to differentiate composites of functions. If $f = v \circ u$, $t = u(x)$ and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist then

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

- Following are some of the standard derivatives (in appropriate domains):

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(e^x\right) = e^x \qquad \frac{d}{dx}\left(\log x\right) = \frac{1}{x}$$

- Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = [u(x)]^{v(x)}$. Here both $f(x)$ and $u(x)$ need to be positive for this technique to make sense.
- ***Rolle's Theorem***: If $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$ such that $f(a) = f(b)$, then there exists some $c$ in $(a, b)$ such that $f'(c) = 0$.
- ***Mean Value Theorem***: If $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists some $c$ in $(a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

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