| Interval | Sign of $f'(x)$ | Nature of function |
|---|---|---|
| $\left[0, \frac{\pi}{4}\right)$ | $> 0$ | $f$ is strictly increasing |
| $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ | $< 0$ | $f$ is strictly decreasing |
| $\left(\frac{5\pi}{4}, 2\pi\right]$ | $> 0$ | $f$ is strictly increasing |

## EXERCISE 6.2

**1.** Show that the function given by $f(x) = 3x + 17$ is strictly increasing on **R**.

**2.** Show that the function given by $f(x) = e^{2x}$ is strictly increasing on **R**.

**3.** Show that the function given by $f(x) = \sin x$ is

(a) strictly increasing in $\left(0, \frac{\pi}{2}\right)$ (b) strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(c) neither increasing nor decreasing in $(0, \pi)$

**4.** Find the intervals in which the function $f$ given by $f(x) = 2x^2 - 3x$ is

(a) strictly increasing (b) strictly decreasing

**5.** Find the intervals in which the function $f$ given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is

(a) strictly increasing (b) strictly decreasing

**6.** Find the intervals in which the following functions are strictly increasing or decreasing:

(a) $x^2 + 2x - 5$ (b) $10 - 6x - 2x^2$

(c) $-2x^3 - 9x^2 - 12x + 1$ (d) $6 - 9x - x^2$

(e) $(x + 1)^3 (x - 3)^3$

**7.** Show that $y = \log(1 + x) - \dfrac{2x}{2 + x}$, $x > -1$, is an increasing function of $x$ throughout its domain.

**8.** Find the values of $x$ for which $y = [x(x - 2)]^2$ is an increasing function.

**9.** Prove that $y = \dfrac{4\sin\theta}{(2 + \cos\theta)} - \theta$ is an increasing function of $\theta$ in $\left[0, \frac{\pi}{2}\right]$.

Similarly, if $c$ is a point of local minima of $f$, then the graph of $f$ around $c$ will be as shown in Fig 6.14(b). Here $f$ is decreasing (i.e., $f'(x) < 0$) in the interval $(c - h, c)$ and increasing (i.e., $f'(x) > 0$) in the interval $(c, c + h)$. This again suggest that $f'(c)$ must be zero.

The above discussion lead us to the following theorem (without proof).

**Theorem 2** Let $f$ be a function defined on an open interval I. Suppose $c \in$ I be any point. If $f$ has a local maxima or a local minima at $x = c$, then either $f'(c) = 0$ or $f$ is not differentiable at $c$.

***Remark*** The converse of above theorem need not be true, that is, a point at which the derivative vanishes need not be a point of local maxima or local minima. For example, if $f(x) = x^3$, then $f'(x) = 3x^2$ and so $f'(0) = 0$. But 0 is neither a point of local maxima nor a point of local minima (Fig 6.15).

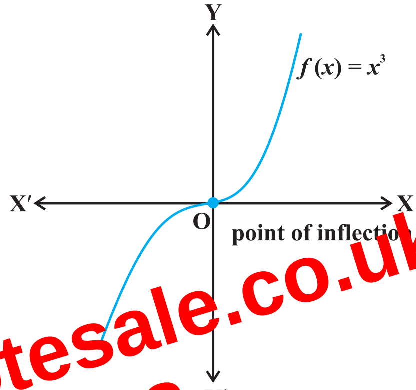


**Fig 6.15**

**Note** A point c in the domain of a function $f$ at which either $f'(c) = 0$ or $f$ is not differentiable is called a *critical point* of $f$. Note that if $f$ is continuous at $c$ and $f'(c) = 0$, then there exists an $h > 0$ such that $f$ is differentiable in the interval $(c - h, c + h)$.

We shall now give a working rule for finding points of local maxima or points of local minima using only the first order derivatives.



**Theorem 3 (First Derivative Test)** Let $f$ be a function defined on an open interval I. Let $f$ be continuous at a critical point $c$ in I. Then

(i) If $f'(x)$ changes sign from positive to negative as $x$ increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of $c$, and $f'(x) < 0$ at every point sufficiently close to and to the right of $c$, then $c$ is a point of *local maxima*.

(ii) If $f'(x)$ changes sign from negative to positive as $x$ increases through $c$, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of $c$, and $f'(x) > 0$ at every point sufficiently close to and to the right of $c$, then $c$ is a point of *local minima*.

(iii) If $f'(x)$ does not change sign as $x$ increases through $c$, then $c$ is neither a point of local maxima nor a point of local minima. Infact, such a point is called *point of inflection* (Fig 6.15).

> **Note** If $c$ is a point of local maxima of $f$, then $f(c)$ is a local maximum value of $f$. Similarly, if $c$ is a point of local minima of $f$, then $f(c)$ is a local minimum value of $f$.

Figures 6.15 and 6.16, geometrically explain Theorem 3.

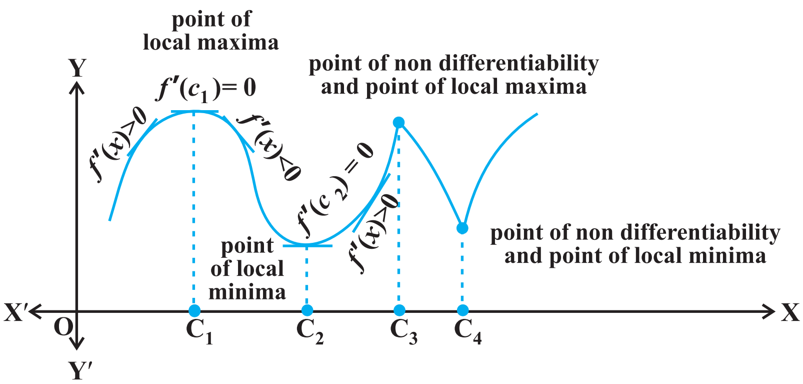


**Fig 6.16**

**Example 29** Find all points of local maxima and local minima of the function $f$ given by

$$f(x) = x^3 - 3x + 3.$$

**Solution** We have

$$f(x) = x^3 - 3x + 3$$

or $$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$$

or $$f'(x) = 0 \text{ at } x = 1 \text{ and } x = -1$$

Thus, $x = \pm 1$ are the only critical points which could possibly be the points of local maxima and/or local minima of $f$. Let us first examine the point $x = 1$.

Note that for values close to 1 and to the right of 1, $f'(x) > 0$ and for values close to 1 and to the left of 1, $f'(x) < 0$. Therefore, by first derivative test, $x = 1$ is a point of local minima and local minimum value is $f(1) = 1$. In the case of $x = -1$, note that $f'(x) > 0$, for values close to and to the left of $-1$ and $f'(x) < 0$, for values close to and to the right of $-1$. Therefore, by first derivative test, $x = -1$ is a point of local maxima and local maximum value is $f(-1) = 5$.

| Values of $x$ | | Sign of $f'(x) = 3(x-1)(x+1)$ |
|---|---|---|
| Close to 1 | to the right (say 1.1 etc.) | $>0$ |
| | to the left (say 0.9 etc.) | $<0$ |
| Close to $-1$ | to the right (say $-0.9$ etc.) | $<0$ |
| | to the left (say $-1.1$ etc.) | $>0$ |

CQ on AB. Let AP = $x$ cm. Note that $\Delta$APD ~ $\Delta$BQC. Therefore, QB = $x$ cm. Also, by Pythagoras theorem, DP = QC = $\sqrt{100-x^2}$ . Let A be the area of the trapezium. Then

$$A \equiv A(x) = \frac{1}{2}(\text{sum of parallel sides})\,(\text{height})$$

$$= \frac{1}{2}(2x+10+10)\left(\sqrt{100-x^2}\right)$$

$$= (x+10)\left(\sqrt{100-x^2}\right)$$

or
$$A'(x) = (x+10)\frac{(-2x)}{2\sqrt{100-x^2}} + \left(\sqrt{100-x^2}\right)$$

$$= \frac{-2x^2-10x+100}{\sqrt{100-x^2}}$$

Now $\quad A'(x) = 0$ gives $2x^2 + 10x - 100 = 0$, i.e., $x = 5$ and $x = -10$.

Since $x$ represents distance, it can not be negative.

So, $\quad x = 5$. Now

$$A''(x) = \frac{\sqrt{100-x^2}\,(-4x-10) - (-2x^2-10x+100)\dfrac{(-2x)}{2\sqrt{100-x^2}}}{100-x^2}$$

$$= \frac{2x^3-300x-1000}{(100-x^2)^{\frac{3}{2}}} \quad \text{(on simplification)}$$

or
$$A''(5) = \frac{2(5)^3-300(5)-1000}{(100-(5)^2)^{\frac{3}{2}}} = \frac{-2250}{75\sqrt{75}} = \frac{-30}{\sqrt{75}} < 0$$

Thus, area of trapezium is maximum at $x = 5$ and the area is given by

$$A(5) = (5+10)\sqrt{100-(5)^2} = 15\sqrt{75} = 75\sqrt{3}\ \text{cm}^2$$

**Example 38** Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

**Solution** Let OC = $r$ be the radius of the cone and OA = $h$ be its height. Let a cylinder with radius OE = $x$ inscribed in the given cone (Fig 6.20). The height QE of the cylinder is given by

$$\frac{QE}{OA} = \frac{EC}{OC} \quad \text{(since } \Delta QEC \sim \Delta AOC\text{)}$$

or $$\frac{QE}{h} = \frac{r-x}{r}$$

or $$QE = \frac{h(r-x)}{r}$$

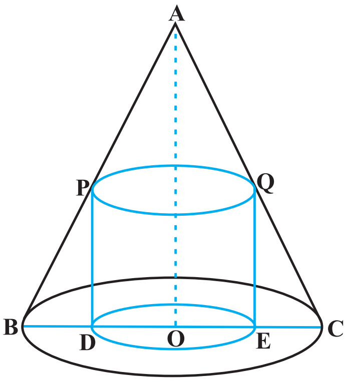

**Fig 6.20**

Let S be the curved surface area of the given cylinder. Then

$$S \equiv S(x) = \frac{2\pi x\, h(r-x)}{r} = \frac{2\pi h}{r}(rx - x^2)$$

or $$\begin{cases} S'(x) = \dfrac{2\pi h}{r}(r-2x) \\ S''(x) = \dfrac{-4\pi h}{r} \end{cases}$$

Now $S'(x) = 0$ gives $x = \frac{r}{2}$. Since $S''(x) < 0$ for all $x$, $S''\left(\frac{r}{2}\right) < 0$. So $x = \frac{r}{2}$ is a point of maxima of S. Hence, the radius of the cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.



### 6.6.1 *Maximum and Minimum Values of a Function in a Closed Interval*

Let us consider a function $f$ given by

$$f(x) = x + 2, x \in (0, 1)$$

Observe that the function is continuous on (0, 1) and neither has a maximum value nor has a minimum value. Further, we may note that the function even has neither a local maximum value nor a local minimum value.

However, if we extend the domain of $f$ to the closed interval [0, 1], then $f$ still may not have a local maximum (minimum) values but it certainly does have maximum value $3 = f(1)$ and minimum value $2 = f(0)$. The maximum value 3 of $f$ at $x = 1$ is called *absolute maximum* value (*global maximum* or *greatest value*) of $f$ on the interval [0, 1]. Similarly, the minimum value 2 of $f$ at $x = 0$ is called the *absolute minimum* value (*global minimum* or *least value*) of $f$ on [0, 1].

Consider the graph given in Fig 6.21 of a continuous function defined on a closed interval $[a, d]$. Observe that the function $f$ has a local minima at $x = b$ and local

## *Miscellaneous Exercise on Chapter 6*

**1.** Using differentials, find the approximate value of each of the following:

(a) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$  (b) $(33)^{-\frac{1}{5}}$

**2.** Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$.

**3.** The two equal sides of an isosceles triangle with fixed base $b$ are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base ?

**4.** Find the equation of the normal to curve $x^2 = 4y$ which passes through the point (1, 2).

**5.** Show that the normal at any point $\theta$ to the curve

$x = a\cos\theta + a\,\theta\sin\theta,\ y = a\sin\theta - a\theta\cos\theta$

is at a constant distance from the origin.

**6.** Find the intervals in which the function $f$ given by

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$$

is (i) increasing (ii) decreasing.



**7.** Find the intervals in which the function $f$ given by $f(x) = x^3 + \frac{1}{x^3},\ x \neq 0$ is

(i) increasing  (ii) decreasing.

**8.** Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.

**9.** A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m$^3$. If building of tank costs Rs 70 per sq metres for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

**10.** The sum of the perimeter of a circle and square is $k$, where $k$ is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

Alternatively, if $f'(x) \geq 0$ for each $x$ in $(a, b)$

(b) decreasing on $(a,b)$ if

$$x_1 < x_2 \text{ in } (a, b) \Rightarrow f(x_1) \geq f(x_2) \text{ for all } x_1, x_2 \in (a, b).$$

Alternatively, if $f'(x) \leq 0$ for each $x$ in $(a, b)$

- The equation of the tangent at $(x_0, y_0)$ to the curve $y = f(x)$ is given by

$$y - y_0 = \left.\frac{dy}{dx}\right]_{(x_0, y_0)} (x - x_0)$$

- If $\frac{dy}{dx}$ does not exist at the point $(x_0, y_0)$, then the tangent at this point is parallel to the y-axis and its equation is $x = x_0$.

- If tangent to a curve $y = f(x)$ at $x = x_0$ is parallel to $x$-axis, then $\left.\frac{dy}{dx}\right]_{x=x_0} = 0$.

- Equation of the normal to the curve $y = f(x)$ at a point $(x_0, y_0)$ is given by

$$y - y_0 = \frac{-1}{\left.\frac{dy}{dx}\right]_{(x_0, y_0)}} (x - x_0)$$

- If $\frac{dy}{dx}$ at the point $(x_0, y_0)$ is zero, then equation of the normal is $x = x_0$.

- If $\frac{dy}{dx}$ at the point $(x_0, y_0)$ does not exist, then the normal is parallel to $x$-axis and its equation is $y = y_0$.

- Let $y = f(x)$, $\Delta x$ be a small increment in $x$ and $\Delta y$ be the increment in $y$ corresponding to the increment in $x$, i.e., $\Delta y = f(x + \Delta x) - f(x)$. Then $dy$ given by

$$dy = f'(x)dx \text{ or } dy = \left(\frac{dy}{dx}\right)\Delta x.$$

is a good approximation of $\Delta y$ when $dx = \Delta x$ is relatively small and we denote it by $dy \approx \Delta y$.

- A point $c$ in the domain of a function $f$ at which either $f'(c) = 0$ or $f$ is not differentiable is called a *critical point* of $f$.

- ***First Derivative Test*** Let $f$ be a function defined on an open interval I. Let $f$ be continuous at a critical point $c$ in I. Then

  (i) If $f'(x)$ changes sign from positive to negative as $x$ increases through c, i.e., if $f'(x) > 0$ at every point sufficiently close to and to the left of $c$, and $f'(x) < 0$ at every point sufficiently close to and to the right of $c$, then $c$ is a point of *local maxima*.

  (ii) If $f'(x)$ changes sign from negative to positive as $x$ increases through $c$, i.e., if $f'(x) < 0$ at every point sufficiently close to and to the left of $c$, and $f'(x) > 0$ at every point sufficiently close to and to the right of $c$, then $c$ is a point of *local minima*.

  (iii) If $f'(x)$ does not change sign as $x$ increases through $c$, then $c$ is neither a point of local maxima nor a point of local minima. Infact, such a point is called *point of inflexion*.

- ***Second Derivative Test*** Let $f$ be a function defined on an interval I and $c \in$ I. Let $f$ be twice differentiable at $c$. Then

  (i) $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$

  The values $f(c)$ is local maximum value of $f$.

  (ii) $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$

  In this case, $f(c)$ is local minimum value of $f$.

  (iii) The test fails if $f'(c) = 0$ and $f''(c) = 0$.

  In this case, we go back to the first derivative test and find whether $c$ is a point of maxima, minima or a point of inflexion.



- Working rule for finding absolute maxima and/or absolute minima

  **Step 1:** Find all critical points of $f$ in the interval, i.e., find points $x$ where either $f'(x) = 0$ or $f$ is not differentiable.

  **Step 2:** Take the end points of the interval.

  **Step 3:** At all these points (listed in Step 1 and 2), calculate the values of $f$.

  **Step 4:** Identify the maximum and minimum values of $f$ out of the values calculated in Step 3. This maximum value will be the absolute maximum value of $f$ and the minimum value will be the absolute minimum value of $f$.