Mechanical vibrations: beats and resonance

Both mechanical vibrations and electric circuits can be accurately modeled by linear differential equations with constant coefficients. Here we will consider only the case of mechanical vibrations, but exactly the same principles apply to electric circuits.

I have posted notes on the algebra and trig underlying some of the analysis.

If u measures the displacement of a mass m from equilibrium, then

$$mu'' + \gamma u' + ku = F. \tag{1}$$

where γ and k are nonnegative constants, and $F = F(t)$ is the external force. The parameter k appears courtesy of Hooke's Law, and is called the "spring constant", even when there is no actual spring.

The damping coefficient γ might be due to friction, internal or external. The term $\gamma u'$ is the nonconservative contribution: it accounts for the energy lost to heat. Often it is the least accurate approximation to the contributing forces, but is accurate enough for many applications to give robust qualitative conclusions about many mechanical systems.

Our strategy to understand the behavior of such systems is to begin with the assumption that there is no damping ($\gamma = 0$) and no forcing ($F = 0$). We then add sinusoidal forcing, where we first encounter the phenomena of "beats" and "resonance". We then consider the effect of damping on this situation.

Undamped free vibrations

A free or unforced vibration is one where $F = 0$. An undamped vibration is one where $\gamma = 0$. We say that the system is *perfectly elastic*. It does not dissipate energy due to frictional or other nonconservative forces.

The general solution to the equation $u'' + ku = 0$ is

$$u = A\cos(\omega_0 t) + B\sin(\omega_0 t), \text{ where } \omega_0 = \sqrt{k/m} \tag{2}$$

and A and B are arbitrary constants. [Check this!] We call ω_0 the *natural harmonic* of the system.

This homogeneous solution can be rewritten in the form

$$u = R\cos(\omega_0 t - \delta), \text{ where } R = \sqrt{A^2 + B^2}, \ R\cos(\delta) = A. \tag{3}$$

The constant R is the amplitude of the wave. The phase shift is δ/ω_0. [Check this!]

Undamped vibrations with periodic forcing

Specificaly, we look at perhaps the simplest periodic forcing, namely $F(t) = \cos(\omega t)$. For our first look at this situation we continue with the assumption that $\gamma = 0$. For simplicity we also take as initial conditions $u(0) = u'(0) = 0$. Other initial conditions will merely effect the contribution from the homogeneous solution, and hence not affect the conclusions.

If $\omega \neq \omega_0$ then

$$u = \frac{F_0/m}{\omega_0^2 - \omega^2}\left[\cos(\omega t) - \cos(\omega_0 t)\right] \tag{4}$$

$$= \frac{F_0/m}{\frac{1}{2}|\omega_0^2 - \omega^2|}\sin(\tfrac{1}{2}|\omega_0 - \omega|\,t)\sin(\tfrac{1}{2}|\omega_0 + \omega|\,t). \tag{5}$$

[Check this!] The first sine factor is a lower frequencey "envelope" which modulates the higher-frequency second sine factor. This is the phenomonon of *beats*. As $\omega \to \omega_0$ the solution changes to

$$u = \frac{F_0/m}{2\omega_0}\,t\,\sin(\omega_0 t). \tag{6}$$

[Check this!] Now the amplitude grows without bound. Thus is the phenomenom called *resonance*.