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A FIRST COURSE IN DIFFERENTIAL EQUATIONS

with Modeling Applications

Preview from Lovola Maryn and University page



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stability, Fourier series, Fourier transforms, linear partial differential equations and boundary-value problems, and numerical methods for partial differential equations. For a one semester course, I assume that the students have successfully completed at least two semesters of calculus. Since you are reading this, undoubtedly you have already examined the table of contents for the topics that are covered. You will not find a "suggested syllabus" in this preface; I will not pretend to be so wise as to tell other teachers what to teach. I feel that there is plenty of material here to pick from and to form a course to your liking. The textbook strikes a reasonable balance between the analytical, qualitative, and quantitative approaches to the study of differential equations. As far as my "underlying philosophy" it is this: An undergraduate textbook should be written with the student's understanding kept firmly in mind, which means to me that the material should be presented in a straightforward, readable, and helpful manner, while keeping the level of theory consistent with the notion of a "first course.

For those who are familiar with the previous editions. I would like to mention a few of the improvements made in this edition.

- Eight new projects appear at the beginning of the book. Each project includes a related problem set, and a correlation of the project material with a section in the text.
- Many exercise sets have been updated by the addition of new problems— • especially discussion problems-to better test and challenge the students. In like manner, some exercise sets have been improved by sending some problems into retirement.
- Additional examples have been added to many sections.
- Several instructors took the time to e-mail me processing their concerns about my approach to linear first-order differentiar equations. In response, Section 2.3, Linear Equations as teen rewritten with the intent to simplify the discussion
- Chorne 2.3, Entern Equations as teen rewritten with the intent to simplify the discussion.
 This edition to have a new section on Green's functions in Chapter 4 for those enclave extra time in their course to consider this elegant application of variation of prammer 0 the solution of initial-value and boundary-value problems. S Ctor 4.8 is optional and its content does not impact any other section

 $y = A\sin(\omega t + \phi)$ and $y = A\cos(\omega t - \phi)$

in describing simple harmonic motion.

At the request of users of the previous editions, a new section on the review of power series has been added to Chapter 6. Moreover, much of this chapter has been rewritten to improve clarity. In particular, the discussion of the modified Bessel functions and the spherical Bessel functions in Section 6.4 has been greatly expanded.

STUDENT RESOURCES

Student Resource Manual (SRM), prepared by Warren S. Wright and Carol D. Wright (ISBN 9781133491927 accompanies A First Course in Differential Equations with Modeling Applications, Tenth Edition and ISBN 9781133491958 accompanies Differential Equations with Boundary-Value Problems, Eighth Edition), provides important review material from algebra and calculus, the solution of every third problem in each exercise set (with the exception of the Discussion Problems and Computer Lab Assignments), relevant command syntax for the computer algebra systems Mathematica and Maple, lists of important concepts, as well as helpful hints on how to start certain problems.

	-1	0.5	0	0	0	0	0	0	0	0	
$\mathbf{A} = \mathbf{M}^{-1}\mathbf{K} =$	0.5	-1	0.5	0	0	0	0	0	0	0	
	0	0.5	-1	0.5	0	0	0	0	0	0	
	0	0	0.5	-1	0.5	0	0	0	0	0	
	0	0	0	0.5	-1	0.5	0	0	0	0	
	0	0	0	0	0.5	-1	0.5	0	0	0	l
	0	0	0	0	0	0.5	-1	0.5	0	0	
	0	0	0	0	0	0	0.5	-1	0.5	0	
	0	0	0	0	0	0	0	0.5	-1	0.5	
	0	0	0	0	0	0	0	0	0.5	-0.5	

As another example, suppose we have a 10-story building, where each floor has a mass 10000 kg, and each k_i value is 5000 kg/s². Then

The eigenvalues of A are found easily using *Mathematica* or another similar computer package. These values are -1.956, -1.826, -1.623, -1.365, -1.075, -0.777, -0.5, -0.267, -0.099, and -0.011, with corresponding frequencies 1.399, 1.351, 1.274, 1.168, 1.037, 0.881, 0.707, 0.517, 0.315, and 0.105 and periods of oscillation $(2\pi/\omega)$ 4.491, 4.651, 4.932, 5.379, 6.059, 7.132, 8.887, 12.153, 19.947, and 59.840. During a typical earthquake whose period might be in the range of 2 to 3 seconds, this building does not seem to be in any danger of developing resonance. However, if the k values were 10 times as large (multiply A by 10), then, for example, the sixth period would be 2.253 seconds, while the fifth through seventh are all on the order of 2-3 seconds. Such a building is more likely to suffer damage in a typical earthquake of period 2-3 seconds.

Related Problems ale.co.uk 1. Consider Market Sale - the building with the same m and k values as in the first exam-Constant of the own the corresponding with the same *m* and *k* values as in the first example. Write own the corresponding system of differential equations. What are the an earthquake vol diplace the building in danger of destruction?
 Constant of the corresponding system of differential equations. What are the same *m* and *k* values as in the second number. Write down the corresponding system of differential equations. What are the matrices M, K and A? Find the eigenvalues for A. What remers for the matrices M, K and A? Find the eigenvalues for A. What remers for the matrices M, K and A? Find the eigenvalues for A. What remers for the matrices M, K and A? Find the eigenvalues for A. What remers for the matrices M, K and A? Find the eigenvalues for A. What remers for the matrices M, K and A? Find the eigenvalues for A. What remers for the matrices M, K and A? Find the eigenvalues for A. What remers for the matrices M, K and A? Find the eigenvalues for A. What remers for the matrices M, K and A? Find the eigenvalues for A. What remers for the matrices M, K and A? Find the eigenvalues for A. What remers for the matrices M, K and A? Find the eigenvalues for A. What remers for the matrices M, K and A? Find the eigenvalues for A. What remers for the matrices M, K and K an

are the matrices M, K, and A? Find the eigenvalues for A. What range of frequencies of an earthquake would place the building in danger of destruction?

- 3. Consider the tallest building on your campus. Assume reasonable values for the mass of each floor and for the proportionality constants between floors. If you have trouble coming up with such values, use the ones in the example problems. Find the matrices M, K, and A, and find the eigenvalues of A and the frequencies and periods of oscillation. Is your building safe from a modest-sized period-2 earthquake? What if you multiplied the matrix **K** by 10 (that is, made the building stiffer)? What would you have to multiply the matrix **K** by in order to put your building in the danger zone?
- 4. Solve the earthquake problem for the three-story building of Problem 1:

$\mathbf{M}\mathbf{X}'' = \mathbf{K}\mathbf{X} + \mathbf{F}(t),$

where $\mathbf{F}(t) = \mathbf{G} \cos \gamma t$, $\mathbf{G} = E\mathbf{B}$, $\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, E = 10,000 lbs is the amplitude of the earthquake force acting at ground level, and $\gamma = 3$ is the frequency of the earthquake (a typical earthquake frequency). See Section 8.3 for the method of solving nonhomogeneous matrix differential equations. Use initial conditions for a building at rest.

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Note, too, that in Example 3 each differential equation possesses the constant solution y = 0, $-\infty < x < \infty$. A solution of a differential equation that is identically zero on an interval *I* is said to be a **trivial solution**.

Solution Curve The graph of a solution ϕ of an ODE is called a **solution curve.** Since ϕ is a differentiable function, it is continuous on its interval *I* of definition. Thus there may be a difference between the graph of the *function* ϕ and the graph of the *solution* ϕ . Put another way, the domain of the function ϕ need not be the same as the interval *I* of definition (or domain) of the solution ϕ . Example 4 illustrates the difference.

EXAMPLE 4 Function versus Solution

(a) function $y = 1/x, x \neq 0$



FIGURE 1.1. In Example 4 the function y = 1/x is not the same as the solution y = 1/x

The domain of y = 1/x, considered simply as a *function*, is the set of all real numbers *x* except 0. When we graph y = 1/x, we plot points in the *xy*-plane corresponding to a judicious sampling of numbers taken from its domain. The rational function y = 1/x is discontinuous at 0, and its graph, in a neighborhood of the origin, is given in Figure 1.1.1(a). The function y = 1/x is not differentiable at x = 0, since the *y*-axis (whose equation is x = 0) is a vertical asymptote of the graph.

Now y = 1/x is also a solution of the linear first-order differential equation xy' + y = 0. (Verify.) But when we say that y = 1/x is a *solution* of this DE, we mean that it is a function defined on an interval *I* on what it is differentiable and satisfies the equation. In other words, y = 1/x is a solution of the DE on *any* interval that does not contain 0, such at $(-3, -1), (-2, 10), (-\infty, 0)$, or $(0, \infty)$. Because the solution curves defined by y = 1/x for -3 < x < -1 and $\frac{1}{2} < x < 10$ are simply segments, or pieces of the solution curves defined by y = 1/x for $-\infty < x < 0$ and 0 < x that appectively, it males sense to take the interval *I* to be as large as possible. Thus we take to be prime $(-\infty, 0)$ or $(0, \infty)$. The solution curve on $(0, \infty)$ when in Figure (1.11b).

Explicit and Implicit Solutions You should be familiar with the terms *explicit functions* and *implicit functions* from your study of calculus. A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an **explicit solution.** For our purposes, let us think of an explicit solution as an explicit formula $y = \phi(x)$ that we can manipulate, evaluate, and differentiate using the standard rules. We have just seen in the last two examples that $y = \frac{1}{16}x^4$, $y = xe^x$, and y = 1/x are, in turn, explicit solutions of $dy/dx = xy^{1/2}$, y'' - 2y' + y = 0, and xy' + y = 0. Moreover, the trivial solution y = 0 is an explicit solution of all three equations. When we get down to the business of actually solving some ordinary differential equations, you will see that methods of solution do not always lead directly to an explicit solution $y = \phi(x)$. This is particularly true when we attempt to solve nonlinear first-orde differential equations. Often we have to be content with a relation or expression G(x, y) = 0 that defines a solution ϕ implicitly.

DEFINITION 1.1.3 Implicit Solution of an ODE

A relation G(x, y) = 0 is said to be an **implicit solution** of an ordinary differential equation (4) on an interval *I*, provided that there exists at least one function ϕ that satisfies the relation as well as the differential equation on *I*.



refers to explicit solutions that are expressible in terms of *elementary* (or familiar) *functions:* finite combinations of integer powers of x, roots, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions.

(*vi*) If every solution of an *n*th-order ODE $F(x, y, y', \dots, y^{(n)}) = 0$ on an interval *I* can be obtained from an *n*-parameter family $G(x, y, c_1, c_2, ..., c_n) = 0$ by appropriate choices of the parameters c_i , i = 1, 2, ..., n, we then say that the family is the general solution of the DE. In solving linear ODEs, we shall impose relatively simple restrictions on the coefficients of the equation; with these restrictions one can be assured that not only does a solution exist on an interval but also that a family of solutions yields all possible solutions. Nonlinear ODEs, with the exception of some first-order equations, are usually difficult or impossible to solve in terms of elementary functions. Furthermore, if we happen to obtain a family of solutions for a nonlinear equation, it is not obvious whether this family contains all solutions. On a practical level, then, the designation "general solution" is applied only to linear ODEs. Don't be concerned about this concept at this point, but store the words "general solution" in the back of your mind-we will come back to this notion in Section 2.3 and again in Chapter 4.

Answers to selected odd-numbered p o

EXERCISES 1.1

- 4. $\frac{d^2u}{dr^2} + \frac{du}{dr} + u = \cos(r+u)$
- 5. $\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$
- $6. \ \frac{d^2R}{dt^2} = -\frac{k}{R^2}$ 7 $(\sin \theta)v''' - (\cos \theta)v' = 2$

7.
$$(\sin \theta)y$$
 $(\cos \theta)y = 2$

8.
$$\ddot{x} - \left(1 - \frac{\dot{x}^2}{3}\right)\dot{x} + x = 0$$

In Problems 9 and 10 determine whether the given first-order differential equation is linear in the indicated dependent variable by matching it with the first differential equation given in (7).

9.
$$(y^2 - 1) dx + x dy = 0$$
; in y; in x

10. $u dv + (v + uv - ue^{u}) du = 0$; in v; in u

In Problem 6 14 verify that the indicated function is an appropriate interval *I* of definition for each solution 1. $(1 - x)y'' - 4xy' + 5y = \cos x$ **FO** 2. $x\frac{d^3y}{d^3} - \left(\frac{dy}{d^3}\right)^4 + y = 0$ 3. $t^5y^{(4)} - t^3y'' + 6y = 0$ 4. $d^2u - d^u$

nyns begin on page ANS-1.

14.
$$y'' + y = \tan x$$
; $y = -(\cos x)\ln(\sec x + \tan x)$

In Problems 15-18 verify that the indicated function $y = \phi(x)$ is an explicit solution of the given first-orde differential equation. Proceed as in Example 2, by considering ϕ simply as a *function*, give its domain. Then by considering ϕ as a *solution* of the differential equation, give at least one interval I of definition

15.
$$(y - x)y' = y - x + 8; \quad y = x + 4\sqrt{x} + 2$$

16. $y' = 25 + y^2; \quad y = 5 \tan 5x$
17. $y' = 2xy^2; \quad y = 1/(4 - x^2)$
18. $2y' = y^3 \cos x; \quad y = (1 - \sin x)^{-1/2}$

In Problems 19 and 20 verify that the indicated expression is an implicit solution of the given first-order differential equation. Find at least one explicit solution $y = \phi(x)$ in each case.



(a) function defined for all *x* except $x = \pm 1$



(**b**) solution defined on interval containing x = 0







See Problems 3–6 in Exercises 1.2 for a continuation of Example 2.

EXAMPLE 3 Second-Order IVP

In Example 7 of Section 1.1 we saw that $x = c_1 \cos 4t + c_2 \sin 4t$ is a two-parameter family of solutions of x'' + 16x = 0. Find a solution of the initial-value problem

$$x'' + 16x = 0, \quad x\left(\frac{\pi}{2}\right) = -2, \quad x'\left(\frac{\pi}{2}\right) = 1.$$
 (4)

SOLUTION We first apply $x(\pi/2) = -2$ to the given family of solutions: $c_1 \cos 2\pi + c_2 \sin 2\pi = -2$. Since $\cos 2\pi = 1$ and $\sin 2\pi = 0$, we find that $c_1 = -2$. We next apply $x'(\pi/2) = 1$ to the one-parameter family $x(t) = -2 \cos 4t + c_2 \sin 4t$. Differentiating and then setting $t = \pi/2$ and x' = 1 gives $8 \sin 2\pi + 4c_2 \cos 2\pi = 1$, from which we see that $c_2 = \frac{1}{4}$. Hence $x = -2 \cos 4t + \frac{1}{4} \sin 4t$ is a solution of (4).

Existence and Uniqueness Two fundamental questions arise in considering an initial-value problem:



for the solution
$$dy/dx = f(x, y)$$
 possess solutions?
whe solution curves pass through the point (x_0, y_0) ?

(When can we be certain that there is precisely one solution curve passing through the point (x_0, y_0) *?*

Note that in Examples 1 and 3 the phrase "*a* solution" is used rather than "*the* solution" of the problem. The indefinite article "a" is used deliberately to suggest the possibility that other solutions may exist. At this point it has not been demonstrated that there is a single solution of each problem. The next example illustrates an initial-value problem with two solutions.

EXAMPLE 4 An IVP Can Have Several Solutions

Each of the functions y = 0 and $y = \frac{1}{16}x^4$ satisfies the differential equation $dy/dx = xy^{1/2}$ and the initial condition y(0) = 0, so the initial-value problem

$$\frac{dy}{dx} = xy^{1/2}, \quad y(0) = 0$$

has at least two solutions. As illustrated in Figure 1.2.5, the graphs of both functions, shown in red and blue pass through the same point (0, 0).

Within the safe confine of a formal course in differential equations one can be fairly confiden that *most* differential equations will have solutions and that solutions of initial-value problems will *probably* be unique. Real life, however, is not so idyllic. Therefore it is desirable to know in advance of trying to solve an initial-value problem



FIGURE 1.2.5 Two solutions curves of the same IVP in Example 4

For the first

medium, and dT/dt the rate at which the temperature of the body changes, then Newton's law of cooling/warming translates into the mathematical statement

$$\frac{dT}{dt} \propto T - T_m$$
 or $\frac{dT}{dt} = k(T - T_m),$ (3)

where k is a constant of proportionality. In either case, cooling or warming, if T_m is a constant, it stands to reason that k < 0.

Spread of a Disease A contagious disease—for example, a flu virus—is spread throughout a community by people coming into contact with other people. Let x(t) denote the number of people who have contracted the disease and y(t) denote the number of people who have not yet been exposed. It seems reasonable to assume that the rate dx/dt at which the disease spreads is proportional to the number of encounters, or interactions, between these two groups of people. If we assume that the number of interactions is jointly proportional to x(t) and y(t)—that is, proportional to the product xy—then

$$\frac{dx}{dt} = kxy,\tag{4}$$

where k is the usual constant of proportionality. Suppose a small community has a fixed population of n people. If one infected person is introduced into this community, then it could be argued that x(t) and y(t) are related by x + y = n + 1. Using this last equation to eliminate y in (4) gives us the model

$$\frac{dx}{dt} = kx(n+1-x).$$
(5)

An obvious initial condition accompanying equation (5) is x(0) = 1.

by the difference equation (1), is said to be a **first-orde reaction**. In chemistry a two reactions followed it same empirical law: If the molecules of substance A decompose into same an obscules, it is a natural assumption that the rate at which this decomposition takes place is proportional to the amount of the first substance that has not undergone conversion; that is, if $Y(t) = t^{t}$ remaining at any time the chloride, (CH₃)₃CCl, into *t*-butyl alcohol, (CH₃)₃COH:

$$(CH_3)_3CCl + NaOH \rightarrow (CH_3)_3COH + NaCl.$$

Only the concentration of the *t*-butyl chloride controls the rate of reaction. But in the reaction

$$CH_3Cl + NaOH \rightarrow CH_3OH + NaCl$$

one molecule of sodium hydroxide, NaOH, is consumed for every molecule of methyl chloride, CH₃Cl, thus forming one molecule of methyl alcohol, CH₃OH, and one molecule of sodium chloride, NaCl. In this case the rate at which the reaction proceeds is proportional to the product of the remaining concentrations of CH₃Cl and NaOH. To describe this second reaction in general, let us suppose one molecule of a substance A combines with one molecule of a substance B to form one molecule of a substance C. If X denotes the amount of chemical C formed at time t and if α and β are, in turn, the amounts of the two chemicals A and B at t = 0 (the initial amounts), then the instantaneous amounts of A and B not converted to chemical C are $\alpha - X$ and $\beta - X$, respectively. Hence the rate of formation of C is given by

$$\frac{dX}{dt} = k(\alpha - X)(\beta - X), \tag{6}$$

where k is a constant of proportionality. A reaction whose model is equation (6) is said to be a second-order reaction.

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35. Falling Body In Problem 23, suppose r = R + s, where *s* is the distance from the surface of the Earth to the falling body. What does the differential contained in Problem 23 become when *s* is my small in comparison to *R*? [*Hitt*::Then all problem largers for $R^{-2}(1 + s/R)^{-2}$

CHAPTER 1 IN REVIEW

In Problems 1 and 2 fill in the blank and then write this result as a linear first-order differential equation that is free of the symbol c_1 and has the form dy/dx = f(x, y). The symbol c_1 represents a constant.

1.
$$\frac{d}{dx}c_1e^{10x} =$$

2. $\frac{d}{dx}(5 + c_1e^{-2x}) =$ _____

In Problems 3 and 4 fill in the blank and then write this result as a linear second-order differential equation that is free of the symbols c_1 and c_2 and has the form F(y, y'') = 0. The symbols c_1, c_2 , and k represent constants.

3. $\frac{d^2}{dx^2}(c_1 \cos kx + c_2 \sin kx) =$ _____ **4.** $\frac{d^2}{dx^2}(c_1 \cosh kx + c_2 \sinh kx) =$ _____

- **36.** Raindrops Keep Falling In meteorology the term *virga* refers to falling raindrops or ice particles that evaporate before they reach the ground. Assume that a typical raindrop is spherical. Starting at some time, which we can designate as t = 0, the raindrop of radius r_0 falls from rest from a cloud and begins to evaporate.
 - (a) If it is assumed that a raindrop evaporates in such a manner that its shape remains spherical, then it also makes sense to assume that the rate at which the raindrop evaporates—that is, the rate at which it loses mass—is proportional to its surface area. Show that this latter assumption implies that the rate at which the radius *r* of the raindrop decreases is a constant. Find *r*(*t*). [*Hint*: See Problem 51 in Exercises 1.1.]
 - (b) If the positive direction is downward, construct a mathematical model for the velocity *v* of the falling raindrop at time *t* > 0. Ignore air resistance. [*Hint*: Use the form of Newton's second law given in (17).]
- **37. Let It Snow** The "snowplow problem" is a classic and appears in many differential equations texts, but it was probably made famous by Ralph Palmer Agnew:

One day it started snowing at a heavy and steady rate. A snowploy parted out at noon, going 2 miles the first hour are to mile the second hou. What time dat it was snowing?

Find the textbook *Differential Equations*, Ralph Palmer Agnew, McGraw-Hill Book Co., and then discuss the Instruction and solution of the mathematical model.

8. Reread this section and classify each mathematical model as linear or nonlinear.

Answers to selected odd-numbered problems begin on page ANS-1.

In Problems 5 and 6 compute y' and y'' and then combine these derivatives with y as a linear second-order differential equation that is free of the symbols c_1 and c_2 and has the form F(y, y' y'') = 0. The symbols c_1 and c_2 represent constants.

5.
$$y = c_1 e^x + c_2 x e^x$$
 6. $y = c_1 e^x \cos x + c_2 e^x \sin x$

In Problems 7–12 match each of the given differential equations with one or more of these solutions:

(a)
$$y = 0$$
, (b) $y = 2$, (c) $y = 2x$, (d) $y = 2x^2$.
7. $xy' = 2y$
8. $y' = 2$
9. $y' = 2y - 4$
10. $xy' = y$
11. $y'' + 9y = 18$
12. $xy'' - y' = 0$

In Problems 13 and 14 determine by inspection at least one solution of the given differential equation.

13. y'' = y' **14.** y' = y(y - 3)

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- Since dy/dx = f(y(x)) is either positive or negative in a subregion R_i , i = 1, 2, 3, a solution y(x) is strictly monotonic—that is, y(x) is either increasing or decreasing in the subregion R_i . Therefore y(x) cannot be oscillatory, nor can it have a relative extremum (maximum or minimum). See Problem 33 in Exercises 2.1.
- If y(x) is *bounded above* by a critical point c_1 (as in subregion R_1 where $y(x) = c_1$ for all x), then the graph of y(x) must approach the graph of the equilibrium solution $y(x) = c_1$ either as $x \to \infty$ or as $x \to -\infty$. If y(x) is *bounded*—that is, bounded above and below by two consecutive critical points (as in subregion R_2 where $c_1 = y(x) = c_2$ for all x)—then the graph of y(x) must approach the graphs of the equilibrium solutions $y(x) = c_1$ and $y(x) = c_2$, one as $x \to \infty$ and the other as $x \to -\infty$. If y(x) is *bounded below* by a critical point (as in subregion R_3 where $c_2 = y(x)$ for all x), then the graph of y(x) must approach the graph of the equilibrium solution $y(x) = c_2$ either as $x \to \infty$ or as $x \to -\infty$. See Problem 34 in Exercises 2.1.

With the foregoing facts in mind, let us reexamine the differential equation in Example 3.

EXAMPLE 4 Example 3 Revisited

The three intervals determined on the *P*-axis or phase line by the critical points 0 and a/b now correspond in the *tP*-plane to three subregions defined by:

P $decreasing P_0$ P R_3 R_4 P R_4 R_4

FIGURE 2.1.7 Phase portrait and solution curves in Example 4

 $R_1: -\infty < P < 0$, $R_2: 0 < P < a/b$, and $A_2: a/b < P < \infty$, where $-\infty$ $t < \infty$. The phase portrait in Figure (2) 17 errs us that P(t) is decreasing in R_1 , increasing in R_2 , and decreasine P_3 . If $P(0) = P_0$ is an initial value, then in R_1, R_2 , and R_3 we have A_2 is tracky, the following:

Let $P_t(O, P(t))$ is bound anabove. Since P(t) is decreasing, P(t) decreases without hour effort increasing t, and so $P(t) \rightarrow 0$ as $t \rightarrow -\infty$. This means that the negative *t*-axis, the graph of the equilibrium solution P(t) = 0, is a horizontal asymptote for a solution curve.

For $0 P_0 a/b$, P(t) is bounded. Since P(t) is increasing, $P(t) \rightarrow a/b$ as $t \rightarrow \infty$ and $P(t) \rightarrow 0$ as $t \rightarrow -\infty$. The graphs of the two equilibrium solutions, P(t) = 0 and P(t) = a/b, are horizontal lines that are horizontal asymptotes for any solution curve starting in this subregion.

(*iii*) For $P_0 > a/b$, P(t) is bounded below. Since P(t) is decreasing, $P(t) \rightarrow a/b$ as $t \rightarrow \infty$. The graph of the equilibrium solution P(t) = a/b is a horizontal asymptote for a solution curve.

In Figure 2.1.7 the phase line is the *P*-axis in the *tP*-plane. For clarity the original phase line from Figure 2.1.5 is reproduced to the left of the plane in which the subregions R_1 , R_2 , and R_3 are shaded. The graphs of the equilibrium solutions P(t) = a/b and P(t) = 0 (the *t*-axis) are shown in the figure as blue dashed lines; the solid graphs represent typical graphs of P(t) illustrating the three cases just discussed.

In a subregion such as R_1 in Example 4, where P(t) is decreasing and unbounded below, we must necessarily have $P(t) \rightarrow -\infty$. Do *not* interpret this last statement to mean $P(t) \rightarrow -\infty$ as $t \rightarrow \infty$; we could have $P(t) \rightarrow -\infty$ as $t \rightarrow T$, where T > 0 is a finite number that depends on the initial condition $P(t_0) = P_0$. Thinking in dynamic terms, P(t) could "blow up" in finite time; thinking graphically, P(t) could have a vertical asymptote at t = T > 0. A similar remark holds for the subregion R_3 .

The differential equation $dy/dx = \sin y$ in Example 2 is autonomous and has an infinite number of critical points, since $\sin y = 0$ at $y = n\pi$, *n* an integer. Moreover, we now know that because the solution y(x) that passes through $(0, -\frac{3}{2})$ is bounded

(ii) Occasionally, a first-order differential equation is not linear in one variable but in linear in the other variable. For example, the differential equation

$$\frac{dy}{dx} = \frac{1}{x + y^2}$$

is not linear in the variable y. But its reciprocal

$$\frac{dx}{dy} = x + y^2$$
 or $\frac{dx}{dy} - x = y^2$

is recognized as linear in the variable x. You should verify that the integrating factor $e^{\int (-1)dy} = e^{-y}$ and integration by parts yield the explicit solution $x = -y^2 - 2y - 2 + ce^y$ for the second equation. This expression is then an implicit solution of the first equation

(iii) Mathematicians have adopted as their own certain words from engineering, which they found appropriately descriptive. The word transient, used earlier, is one of these terms. In future discussions the words input and output will occasionally pop up. The function f in (2) is called the **input** or **driving func**tion; a solution y(x) of the differential equation for a given input is called the output or response.

(iv) The term special functions mentioned in conjunction with the error function also applies to the sine integral function and the Fresnel sine integral introduced in Problems 55 and 56 in Exercises 2.3. "Special Functions" is actually a well-defined branch of mathematics. More special functions are

- **3.** $\frac{dy}{dx} + y = e^{3x}$ **4.** $3\frac{dy}{dx} + 12y = 4$
- **5.** $y' + 3x^2y = x^2$ **6.** $y' + 2xy = x^3$
- 8. $y' = 2y + x^2 + 5$ 7. $x^2y' + xy = 1$ **9.** $x \frac{dy}{dx} - y = x^2 \sin x$ **10.** $x \frac{dy}{dx} + 2y = 3$

17. $\cos x \frac{dy}{dx} + (\sin x)y = 1$

- **11.** $x \frac{dy}{dx} + 4y = x^3 x$ **12.** $(1 + x) \frac{dy}{dx} xy = x + x^2$ 13. $x^2y' + x(x+2)y = e^x$ 14. $xy' + (1 + x)y = e^{-x} \sin 2x$ **15.** $y dx - 4(x + y^6) dy = 0$ **16.** $v dx = (ve^y - 2x) dv$
- **EXERCISES 2.3** In Problems 1–24 find the vertex solution of the give (d)² ferential count (e) of which argest interval *I* over which the set also attom is defined. Determine, whether there are any transient terms in the vertex is solution. 1. $\frac{dy}{dx} = 5y$ 2. $\frac{dy}{dx} = 5y$ 3. $\frac{dy}{dx} = 5y$ 4. $\frac{dy}{dx} = 5y$ 5. $\frac{dy}{d$ Are selected odd-numbered problems begin on page ANS-2. **18.** $\cos^2 x \sin x \frac{dy}{dx} + (\cos^3 x)y = 1$ **19.** $(x + 1)\frac{dy}{dx} + (x + 2)y = 2xe^{-x}$ **20.** $(x+2)^2 \frac{dy}{dx} = 5 - 8y - 4xy$ **21.** $\frac{dr}{d\theta} + r \sec \theta = \cos \theta$ **22.** $\frac{dP}{dt} + 2tP = P + 4t - 2$ **23.** $x \frac{dy}{dx} + (3x + 1)y = e^{-3x}$ **24.** $(x^2 - 1)\frac{dy}{dx} + 2y = (x + 1)^2$

In Problems 25-36 solve the given initial-value problem. Give the largest interval I over which the solution is defined

25. $\frac{dy}{dx} = x + 5y, \quad y(0) = 3$ **26.** $\frac{dy}{dx} = 2x - 3y$, $y(0) = \frac{1}{3}$ **27.** $xy' + y = e^x$, y(1) = 2

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point as (x_1, y_1) with (x_0, y_0) in the above discussion, we obtain an approximation $y_2 \approx y(x_2)$ corresponding to two steps of length *h* from x_0 , that is, $x_2 = x_1 + h = x_0 + 2h$, and

$$y(x_2) = y(x_0 + 2h) = y(x_1 + h) \approx y_2 = y_1 + hf(x_1, y_1).$$

Continuing in this manner, we see that y_1, y_2, y_3, \ldots , can be defined recursively by the general formula

$$y_{n+1} = y_n + hf(x_n, y_n),$$
 (3)

where $x_n = x_0 + nh$, n = 0, 1, 2, ... This procedure of using successive "tangent lines" is called **Euler's method.**

EXAMPLE 1 Euler's Method

J

Consider the initial-value problem $y' = 0.1\sqrt{y} + 0.4x^2$, y(2) = 4. Use Euler's method to obtain an approximation of y(2.5) using first h = 0.1 and then h = 0.05.

SOLUTION With the identification $f(x, y) = 0.1\sqrt{y} + 0.4x^2$, (3) becomes

$$y_{n+1} = y_n + h(0.1\sqrt{y_n} + 0.4x_n^2).$$

Then for h = 0.1, $x_0 = 2$, $y_0 = 4$, and n = 0 we fin

$$y_1 = y_0 + h(0.1\sqrt{y_0} + 0.4x_0^2) = 4 + 0.10(1\sqrt{4} + 0.4(2)^2) = 4.18,$$

which, as we have already seen, is in estimate to the value of y(2.1). However, if we use the smaller step size y = 0.05, it takes two steps to reach x = 2.1. From

$$\mathbf{O}_{1} = 4 + 0.25((1)/4 + 0.4(2)^{2}) = 4.09$$

$$\mathbf{O}_{4.09} = 0.05((0.1\sqrt{4.09} + 0.4(2.05)^{2})) = 4.18416187$$

remarke $y_1 \approx y(2.05)$ and $y_2 \approx y(2.1)$. The remainder of the calculations were carried out by using software. The results are summarized in Tables 2.6.1 and 2.6.2, where each entry has been rounded to four decimal places. We see in Tables 2.6.1 and 2.6.2 that it takes five steps with h = 0.1 and 10 steps with h = 0.05, respectively, to get to x = 2.5. Intuitively, we would expect that $y_{10} = 5.0997$ corresponding to h = 0.05 is the better approximation of y(2.5) than the value $y_5 = 5.0768$ corresponding to h = 0.1.

In Example 2 we apply Euler's method to a differential equation for which we have already found a solution. We do this to compare the values of the approximations y_n at each step with the true or actual values of the solution $y(x_n)$ of the initial-value problem.

EXAMPLE 2 Comparison of Approximate and Actual Values

Consider the initial-value problem y' = 0.2xy, y(1) = 1. Use Euler's method to obtain an approximation of y(1.5) using first h = 0.1 and then h = 0.05.

SOLUTION With the identification f(x, y) = 0.2xy, (3) becomes

$$y_{n+1} = y_n + h(0.2x_ny_n)$$

where $x_0 = 1$ and $y_0 = 1$. Again with the aid of computer software we obtain the values in Tables 2.6.3 and 2.6.4 on page 78.

IABLE 2.6. I	h = 0.1
<i>x</i> _n	y_n
2.00	4.0000
2.10	4.1800
2.20	4.3768

4.5914

4.8244

5.0768

TABLE 2.6.2	h = 0.05

2.30

2.40

2.50



					x _n	y_n	Actual value	Abs. error	% Rel. error
					1.00	1.0000	1.0000	0.0000	0.00
					1.05	1.0100	1.0103	0.0003	0.03
ABLE 2	2. 6.3 h =	= 0.1			1.10	1.0206	1.0212	0.0006	0.06
r		Actual value	Abs error	% Pal arror	1.15	1.0318	1.0328	0.0009	0.09
λ_n	<i>yn</i>	Actual value	Abs. choi	70 Kel. elloi	1.20	1.0437	1.0450	0.0013	0.12
1.00	1.0000	1.0000	0.0000	0.00	1.25	1.0562	1.0579	0.0016	0.16
1.10	1.0200	1.0212	0.0012	0.12	1.30	1.0694	1.0714	0.0020	0.19
1.20	1.0424	1.0450	0.0025	0.24	1.35	1.0833	1.0857	0.0024	0.22
1.30	1.0675	1.0714	0.0040	0.37	1.40	1.0980	1.1008	0.0028	0.25
1.40	1.0952	1.1008	0.0055	0.50	1.45	1.1133	1.1166	0.0032	0.29
1.50	1.1259	1.1331	0.0073	0.64	1.50	1.1295	1.1331	0.0037	0.32
1.10 1.20 1.30 1.40 1.50	1.0200 1.0424 1.0675 1.0952 1.1259	1.0212 1.0450 1.0714 1.1008 1.1331	0.0012 0.0025 0.0040 0.0055 0.0073	0.12 0.24 0.37 0.50 0.64	1.30 1.35 1.40 1.45 1.50	1.0694 1.0833 1.0980 1.1133 1.1295	1.0714 1.0857 1.1008 1.1166 1.1331	0.0020 0.0024 0.0028 0.0032 0.0037	

TABLE 2.6.4 *h* = 0.05

In Example 1 the true or actual values were calculated from the known solution $y = e^{0.1(x^2-1)}$. (Verify.) The **absolute error** is defined to b

actual value – approximation.

The relative error and percentage relative error are, in turn,

 $\frac{absolute\ error}{|actual\ value|} \times 100.$ absolute error and

It is apparent from Tables 2.6.3 and 2.6.4 that the accurate of the approximations improves as the step size h decreases. Also we see that even though the percentage relative error is growing with each step it does not appear to be that bad. But you should not be deceived on the Grample. If we simply change the coefficient of the right side $(x_1 \in D)$ is Example 2 from 0.2 to 2, then at $x_n = 1.5$ the percentage relative errors increase dramatically see roblem 4 in Exercises 2.6.

Preview freender to action of the second sec \equiv A Cavear event of the solution of the solution of the solution of a differential equation can be approximated. Although attractive for its simple solution, *Euler's method is seldom used in serious calculations*. It was introduced here simply to give you a first taste of numerical methods. We will go into greater detail in discussing numerical methods that give significantly greater accuracy, notably the **fourth order Runge-Kutta method**, referred to as the **RK4 method**, in Chapter 9.

> \equiv Numerical Solvers Regardless of whether we can actually fin an explicit or implicit solution, if a solution of a differential equation exists, it represents a smooth curve in the Cartesian plane. The basic idea behind any numerical method for first-orde ordinary differential equations is to somehow approximate the y-values of a solution for preselected values of x. We start at a specifie initial point (x_0, y_0) on a solution curve and proceed to calculate in a step-by-step fashion a sequence of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ whose y-coordinates y_i approximate the y-coordinates $y(x_i)$ of points $(x_1, y(x_1)), (x_2, y(x_2)), \dots, (x_n, y(x_n))$ that lie on the graph of the usually unknown solution y(x). By taking the x-coordinates close together (that is, for small values of h) and by joining the points (x_1, y_1) , $(x_2, y_2), \ldots, (x_n, y_n)$ with short line segments, we obtain a polygonal curve whose qualitative characteristics we hope are close to those of an actual solution curve. Drawing curves is something that is well suited to a computer. A computer program written to either implement a numerical method or render a visual representation of an approximate solution curve fittin the numerical data produced by this method is referred to as a **numerical solver.** Many different numerical solvers are commercially available, either embedded in a larger software package, such as a computer

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FIGURE 2.6.3 Comparison of the Runge-Kutta (RK4) and Euler methods

6

5

4

3

algebra system, or provided as a stand-alone package. Some software packages simply plot the generated numerical approximations, whereas others generate hard numerical data as well as the corresponding approximate or **numerical solution curves.** By way of illustration of the connect-the-dots nature of the graphs produced by a numerical solver, the two colored polygonal graphs in Figure 2.6.3 are the numerical solution curves for the initial-value problem y' = 0.2xy, y(0) = 1 on the interval [0, 4] obtained from Euler's method and the RK4 method using the step size h = 1. The blue smooth curve is the graph of the exact solution $y = e^{0.1x^2}$ of the IVP. Notice in Figure 2.6.3 that, even with the ridiculously large step size of h = 1, the RK4 method produces the more believable "solution curve." The numerical solution curve obtained from the RK4 method is indistinguishable from the actual solution curve on the interval [0, 4] when a more typical step size of h = 0.1 is used.

Using a Numerical Solver Knowledge of the various numerical methods is

not necessary in order to use a numerical solver. A solver usually requires that the differential equation be expressed in normal form dy/dx = f(x, y). Numerical solvers that generate only curves usually require that you supply f(x, y) and the initial data x_0 and y_0 and specify the desired numerical method. If the idea is to approximate the numerical value of y(a), then a solver may additionally require that you state a value for h or, equivalently, give the number of steps that you want to take to get from $x = x_0$ to x = a. For example, if we wanted to approximate y(4) for the IVP illustrated in Figure 2.6.3, then, starting at x = 0 it would take four steps to reach x = 4 with a step size of h = 1; 40 steps is equivalent to a step size of h = 0.1. Although we will not delve here into the many problems that one carrier counter when attempting to approximate mathematical quantities, you should at east be aware of the fact that a numerical solver may break down ner costern points or give an incomplete or misleading picture when applied with first-order differential equations in the normal form Figure 2 2 5 states the graph obtained by applying Euler's method to a certake the contrast of the problem dy/dx = f(x, y), y(0) = 1. Equivalent results were obtained using the confident commercial numerical solvers, yet the graph is hardly coplarsh h solution curve. (Why?) There are several avenues of recourse where an mencal solver has difficulties; three of the more obvious are decrease the size, use another numerical method, and try a different numerical solver.

EXERCISES 2.6

In Problems 1 and 2 use Euler's method to obtain a fourdecimal approximation of the indicated value. Carry out the recursion of (3) by hand, first using h = 0.1 and then using h = 0.05.

1.
$$y' = 2x - 3y + 1$$
, $y(1) = 5$; $y(1.2)$
2. $y' = x + y^2$, $y(0) = 0$; $y(0.2)$

In Problems 3 and 4 use Euler's method to obtain a fourdecimal approximation of the indicated value. First use h = 0.1 and then use h = 0.05. Find an explicit solution for each initial-value problem and then construct tables similar to Tables 2.6.3 and 2.6.4.

3.
$$y' = y, y(0) = 1; y(1.0)$$

4. $y' = 2xy, y(1) = 1; y(1.5)$

Answers to selected odd-numbered problems begin on page ANS-3.

In Problems 5–10 use a numerical solver and Euler's method to obtain a four-decimal approximation of the indicated value. First use h = 0.1 and then use h = 0.05.

5.
$$y' = e^{-y}, y(0) = 0; y(0.5)$$

6. $y' = x^2 + y^2, y(0) = 1; y(0.5)$
7. $y' = (x - y)^2, y(0) = 0.5; y(0.5)$
8. $y' = xy + \sqrt{y}, y(0) = 1; y(0.5)$
9. $y' = xy^2 - \frac{y}{x}, y(1) = 1; y(1.5)$
10. $y' = y - y^2, y(0) = 0.5; y(0.5)$

In Problems 11 and 12 use a numerical solver to obtain a numerical solution curve for the given initial-value problem. First use Euler's method and then the RK4 method. Use

Modeling with First-Order **Differential Equations**

- **3.1** Linear Models
- **3.2** Nonlinear Models
- **3.3** Modeling with Systems of First-Order DEs

Chapter 3 in Review



In Section 1.3 we saw how a first-order di ferential equation could be used as a mathematical model in the study of population growth, radioactive decay, continuous compound interest, cooling of bodies, mixtures, chemical reactions, fluid draining from a tank, velocity of a falling bod, and current in a series circuit. Using the methods of Chapter 2, we are now able to solve some of the linear DEs in Section 3.1 and nonlinear DEs in Section 3.2 that commonly appear in applications.

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FIGURE 3.1.2 Growth (k > 0) and decay (k < 0)

Notice in Example 1 that the actual number P_0 of bacteria present at time t = 0played no part in determining the time required for the number in the culture to triple. The time necessary for an initial population of, say, 100 or 1,000,000 bacteria to triple is still approximately 2.71 hours.

As shown in Figure 3.1.2, the exponential function e^{kt} increases as t increases for k > 0 and decreases as t increases for k < 0. Thus problems describing growth (whether of populations, bacteria, or even capital) are characterized by a positive value of k, whereas problems involving decay (as in radioactive disintegration) yield a negative k value. Accordingly, we say that k is either a growth constant (k > 0) or a decay constant (k < 0).

Half-Life In physics the **half-life** is a measure of the stability of a radioactive substance. The half-life is simply the time it takes for one-half of the atoms in an initial amount A_0 to disintegrate, or transmute, into the atoms of another element. The longer the half-life of a substance, the more stable it is. For example, the halflife of highly radioactive radium, Ra-226, is about 1700 years. In 1700 years onehalf of a given quantity of Ra-226 is transmuted into radon, Rn-222. The most commonly occurring uranium isotope, U-238, has a half-life of approximately 4,500,000,000 years. In about 4.5 billion years, one-half of a quantity of U-238 is transmuted into lead, Pb-206.

EXAMPLE 2 Half-Life of Plutonium

A breeder reactor converts relatively stable uranize -238 into the isotope plutonium-239. After 15 years it is determined that 0.943 so if the initial amount A_0 of plutonium has disintegrated. Find the molife of time isotope if the rate of disintegration is pro-SCIENCE Let A(t) le Oe et amount of plutor Example 1 the science of the mitial-value problem DREVIEW B30 dA

Let $A(t) \neq 0$ amount of plutonium remaining at time t. As in

$$\frac{dA}{dt} = kA, \quad A(0) = A_0$$

is $A(t) = A_0 e^{kt}$. If 0.043% of the atoms of A_0 have disintegrated, then 99.957% of the substance remains. To find the decay constant k, we use $0.99957A_0 = A(15)$ —that is, $0.99957A_0 = A_0e^{15k}$. Solving for k then gives $k = \frac{1}{15} \ln 0.99957 = -0.00002867$. Hence $A(t) = A_0e^{-0.00002867t}$. Now the half-life is the corresponding value of time at which $A(t) = \frac{1}{2}A_0$. Solving for t gives $\frac{1}{2}A_0 = A_0e^{-0.00002867t}$, or $\frac{1}{2} = e^{-0.00002867t}$. The last equation vields

$$t = \frac{\ln 2}{0.00002867} \approx 24,180 \text{ yr.}$$

Carbon Dating About 1950, a team of scientists at the University of Chicago led by the chemist Willard Libby devised a method using a radioactive isotope of carbon as a means of determining the approximate ages of carbonaceous fossilized matter. The theory of carbon dating is based on the fact that the radioisotope carbon-14 is produced in the atmosphere by the action of cosmic radiation on nitrogen-14. The ratio of the amount of C-14 to the stable C-12 in the atmosphere appears to be a constant, and as a consequence the proportionate amount of the isotope present in all living organisms is the same as that in the atmosphere. When a living organism dies, the absorption of C-14, by breathing, eating, or photosynthesis, ceases. By comparing the proportionate amount of C-14, say, in a fossil with the constant amount ratio found in the atmosphere, it is possible to obtain a reasonable estimation of its age. The method is based on the knowledge of the half-life of C-14. Libby's calculated

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FIGURE 3.1.3 A page of the Gnostic Gospel of Judas

value of the half-life of C-14 was approximately 5600 years, but today the commonly accepted value of the half-life is approximately 5730 years. For his work, Libby was awarded the Nobel Prize for chemistry in 1960. Libby's method has been used to date wooden furniture found in Egyptian tombs, the woven flax wrappings of the Dead Sea Scrolls, a recently discovered copy of the Gnostic Gospel of Judas written on papyrus, and the cloth of the enigmatic Shroud of Turin. See Figure 3.1.3 and Problem 12 in Exercises 3.1.



A fossilized bone is found to contain 0.1% of its original amount of C-14. Determine the age of the fossil.

SOLUTION The starting point is again $A(t) = A_0 e^{kt}$. To determine the value of the decay constant k we use the fact that $\frac{1}{2}A_0 = A(5730)$ or $\frac{1}{2}A_0 = A_0 e^{5730k}$. The last equation implies $5730k = \ln \frac{1}{2} = -\ln 2$ and so we get $k = -(\ln 2)/5730 = -0.00012097$. Therefore $A(t) = A_0 e^{-0.00012097t}$. With $A(t) = 0.001A_0$ we have $0.001A_0 = A_0 e^{-0.00012097t}$ and $-0.00012097t = \ln(0.001) = -\ln 1000$. Thus

$$t = \frac{\ln 1000}{0.00012097} \approx 57,100 \text{ years.}$$

The date found in Example 3 is really at the border of accuracy for this method. The usual carbon-14 technique is limited to about 10 half-lives of the isotope, or roughly 60,000 years. One reason for this limitation is that the chemical analysis needed to obtain an accurate measurement of the remaining could become somewhat formidable around the point $0.001A_0$. Also mis analysis demands the destruction of a rather large sample of the specimeer forms measurement is accomplished indirectly, based on the tectral residentiation from the specimen, then it is very difficult to distinguishing we are radiation from the specimen and the normal background radiation.^{*} Intercently the usators particle accelerator has enabled scientists to separate the C-14 from the statue C-12 directly. When the precise value of the ratio of C-14 to C14 to C14 the computed, the accuracy can be extended to 70,000 to 100,000 years. Other totopic techniques, such as using potassium-40 and argon-40, can give accord are also sometimes possible.

Newton's Law of Cooling/Warming In equation (3) of Section 1.3 we saw that the mathematical formulation of Newton's empirical law of cooling/warming of an object is given by the linear first-order di ferential equation

$$\frac{dT}{dt} = k(T - T_m),\tag{2}$$

where *k* is a constant of proportionality, T(t) is the temperature of the object for t > 0, and T_m is the ambient temperature—that is, the temperature of the medium around the object. In Example 4 we assume that T_m is constant.

EXAMPLE 4 Cooling of a Cake

When a cake is removed from an oven, its temperature is measured at 300° F. Three minutes later its temperature is 200° F. How long will it take for the cake to cool off to a room temperature of 70° F?

*The number of disintegrations per minute per gram of carbon is recorded by using a Geiger counter. The lower level of detectability is about 0.1 disintegrations per minute per gram.



T(t)	t (min)		
75°	20.1		
74°	21.3		
73°	22.8		
72°	24.9		
71°	28.6		
70.5°	32.3		
(b)			

FIGURE 3.1.4 Temperature of cooling

cake in Example 4

SOLUTION In (2) we make the identification $T_m = 70$. We must then solve the initial-value problem

$$\frac{dT}{dt} = k(T - 70), \quad T(0) = 300 \tag{3}$$

and determine the value of k so that T(3) = 200.

Equation (3) is both linear and separable. If we separate variables,

$$\frac{dT}{T-70} = k \, dt,$$

yields $\ln |T - 70| = kt + c_1$, and so $T = 70 + c_2 e^{kt}$. When t = 0, T = 300, so $300 = 70 + c_2$ gives $c_2 = 230$; therefore $T = 70 + 230e^{kt}$. Finally, the measurement T(3) = 200 leads to $e^{3k} = \frac{13}{23}$, or $k = \frac{1}{3} \ln \frac{13}{23} = -0.19018$. Thus

$$T(t) = 70 + 230e^{-0.19018t}.$$
(4)

We note that (4) furnishes no finite solution to T(t) = 70, since $\lim_{t\to\infty} T(t) = 70$. Yet we intuitively expect the cake to reach room temperature after a reasonably long period of time. How long is "long"? Of course, we should not be disturbed by the fact that the model (3) does not quite live up to our physical intuition. Parts (a) and (b) of Figure 3.1.4 clearly show that the cake will be approximately at room temperature in about one-half hour. _

The ambient temperature in (2) need not be a constant but could be a function $T_m(t)$ of time t. See Problem 18 in Exercises 3.1.

Mixtures The mixing of two fluid some rises gives rise to a linear first-orde differential equation. When we discussed the mixing of two brine solutions in Section 1.3, we assume that the rate A'(t) at which the amount of salt in the mixing

) - (output rate of salt) =
$$R_{in} - R_{out}$$
. (5)

ble 5 we solve equation (8) of Section 1.3.

EXAMPLE 5 Mixture of Two Salt Solutions

Preview from h Page 13 Recall that the large tank considered in Section 1.3 held 300 gallons of a brine solution. Salt was entering and leaving the tank; a brine solution was being pumped into the tank at the rate of 3 gal/min; it mixed with the solution there, and then the mixture was pumped out at the rate of 3 gal/min. The concentration of the salt in the inflo, or solution entering, was 2 lb/gal, so salt was entering the tank at the rate $R_{in} = (2 \text{ lb/gal}) \cdot (3 \text{ gal/min}) = 6 \text{ lb/min}$ and leaving the tank at the rate $R_{out} =$ $(A/300 \text{ lb/gal}) \cdot (3 \text{ gal/min}) = A/100 \text{ lb/min}$. From this data and (5) we get equation (8) of Section 1.3. Let us pose the question: If 50 pounds of salt were dissolved initially in the 300 gallons, how much salt is in the tank after a long time?

> **SOLUTION** To find the amount of salt A(t) in the tank at time t, we solve the initialvalue problem

$$\frac{dA}{dt} + \frac{1}{100}A = 6, \quad A(0) = 50.$$

Note here that the side condition is the initial amount of salt A(0) = 50 in the tank and not the initial amount of liquid in the tank. Now since the integrating factor of the linear differential equation is $e^{t/100}$, we can write the equation as

$$\frac{d}{dt} \left[e^{t/100} A \right] = 6 e^{t/100}$$

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 $\frac{1}{2}$ pound of salt per gallon is pumped into the tank at a rate of 6 gal/min. The well-mixed solution is then pumped out at a slower rate of 4 gal/min. Find the number of pounds of salt in the tank after 30 minutes.

- **28.** In Example 5 the size of the tank containing the salt mixture was not given. Suppose, as in the discussion following Example 5, that the rate at which brine is pumped into the tank is 3 gal/min but that the well-stirred solution is pumped out at a rate of 2 gal/min. It stands to reason that since brine is accumulating in the tank at the rate of 1 gal/min, any finite tank must eventually overflo . Now suppose that the tank has an open top and has a total capacity of 400 gallons.
 - (a) When will the tank overflow
 - (b) What will be the number of pounds of salt in the tank at the instant it overflows
 - (c) Assume that although the tank is overflowing, brine solution continues to be pumped in at a rate of 3 gal/min and the well-stirred solution continues to be pumped out at a rate of 2 gal/min. Devise a method for determining the number of pounds of salt in the tank at t = 150 minutes.
 - (d) Determine the number of pounds of salt in the tank as $t \rightarrow \infty$. Does your answer agree with your intuition?
 - (e) Use a graphing utility to plot the graph of *A*(*t*) on the interval [0, 500).

Series Circuits

- **29.** A 30-volt electromotive force happened to an *LR*-series circuit in which the multiance is 0.1 hence are the resistance P 50 orms. Find the current P_{2} if h(0) = 0. Differmine the current as $h \to \infty$
- **30.** Solve equation (7) under the assumption that $E(t) = E_0 \sin \omega t$ and $i(0) = i_0$.
- **31.** A 100-volt electromotive force is applied to an *RC*-series circuit in which the resistance is 200 ohms and the capacitance is 10^{-4} farad. Find the charge q(t) on the capacitor if q(0) = 0. Find the current i(t).
- **32.** A 200-volt electromotive force is applied to an *RC*-series circuit in which the resistance is 1000 ohms and the capacitance is 5×10^{-6} farad. Find the charge q(t) on the capacitor if i(0) = 0.4. Determine the charge and current at t = 0.005 s. Determine the charge as $t \rightarrow \infty$.
- **33.** An electromotive force

$$E(t) = \begin{cases} 120, & 0 & t & 20\\ 0, & t & > 20 \end{cases}$$

is applied to an *LR*-series circuit in which the inductance is 20 henries and the resistance is 2 ohms. Find the current i(t) if i(0) = 0. **34.** Suppose an *RC*-series circuit has a variable resistor. If the resistance at time *t* is given by $R = k_1 + k_2 t$, where k_1 and k_2 are known positive constants, then (9) becomes

$$(k_1 + k_2 t)\frac{dq}{dt} + \frac{1}{C}q = E(t).$$

If $E(t) = E_0$ and $q(0) = q_0$, where E_0 and q_0 are constants, show that

$$q(t) = E_0 C + (q_0 - E_0 C) \left(\frac{k_1}{k_1 + k_2 t}\right)^{1/Ck_2}$$

Additional Linear Models

35. Air Resistance In (14) of Section 1.3 we saw that a differential equation describing the velocity v of a falling mass subject to air resistance proportional to the instantaneous velocity is

$$m\frac{dv}{dt} = mg - kv,$$

where k > 0 is a constant of proportionality. The positive direction is downward.

- (a) Solve the equation subject to the initial condition $v(0) = v_0$.
- (b) Use the solution in p.r. (a) to determine the limiting, a terrainal, velocity of the mass. We saw how a terraine the terminal velocity without solving the DE in Problem 40 in Exercises 2.1.
 - Let s be tarance s, measured from the point where the mass was released above ground, is related to velocity v by ds/dt = v(t), find an explicit expression for s(t) if s(0) = 0.
- **36.** How High?—No Air Resistance Suppose a small cannonball weighing 16 pounds is shot vertically upward, as shown in Figure 3.1.12, with an initial velocity $v_0 = 300$ ft/s. The answer to the question "How high does the cannonball go?" depends on whether we take air resistance into account.
 - (a) Suppose air resistance is ignored. If the positive direction is upward, then a model for the state of the cannonball is given by $d^2s/dt^2 = -g$ (equation (12) of Section 1.3). Since ds/dt = v(t) the last

FIGURE 3.1.12 Find the maximum height of the cannonball in Problem 36



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15. Air Resistance A differential equation for the velocity v of a falling mass m subjected to air resistance proportional to the square of the instantaneous velocity is

$$m\frac{dv}{dt} = mg - kv^2,$$

where k > 0 is a constant of proportionality. The positive direction is downward.

- (a) Solve the equation subject to the initial condition $v(0) = v_0$.
- (**b**) Use the solution in part (a) to determine the limiting, or terminal, velocity of the mass. We saw how to determine the terminal velocity without solving the DE in Problem 41 in Exercises 2.1.
- (c) If the distance *s*, measured from the point where the mass was released above ground, is related to velocity *v* by ds/dt = v(t), find an explicit expression for *s*(*t*) if *s*(0) = 0.
- 16. How High?—Nonlinear Air Resistance Consider the 16-pound cannonball shot vertically upward in Problems 36 and 37 in Exercises 3.1 with an initial velocity $v_0 = 300$ ft/s. Determine the maximum height attained by the cannonball if air resistance is assumed to be proportional to the square of the instantaneous velocity. Assume that the positive direction is upward and take k = 0.0003. [*Hint*: Slightly modify the DE in Problem 15.]
- 17. That Sinking Feeling (a) Determine a differentiate equation for the velocity v(t) of a mass m similar in water that imparts a resistance proportional to the square of the instantaneous velocity and also exerts an upwind due yant force whose magnitude to exerts a chimedes' principle we problem 18 in Exercises 1.3. Assume that the politive direction is downward.
 - (b) Solve the differential equation in part (a).
 - (c) Determine the limiting, or terminal, velocity of the sinking mass.
- 18. Solar Collector The differential equation

$$\frac{dy}{dx} = \frac{-x + \sqrt{x^2 + y^2}}{y}$$

describes the shape of a plane curve C that will reflect all incoming light beams to the same point and could be a model for the mirror of a reflecting telescope, a satellite antenna, or a solar collector. See Problem 29 in Exercises 1.3. There are several ways of solving this DE.

(a) Verify that the differential equation is homogeneous (see Section 2.5). Show that the substitution y = ux yields

$$\frac{u \, du}{\sqrt{1 + u^2} \left(1 - \sqrt{1 + u^2}\right)} = \frac{dx}{x}.$$

Use a CAS (or another judicious substitution) to integrate the left-hand side of the equation. Show that the curve C must be a parabola with focus at the origin and is symmetric with respect to the *x*-axis.

- (b) Show that the first differential equation can also be solved by means of the substitution $u = x^2 + y^2$.
- **19. Tsunami** (a) A simple model for the shape of a tsunami is given by

$$\frac{dW}{dx} = W\sqrt{4-2W}$$

where W(x) > 0 is the height of the wave expressed as a function of its position relative to a point offshore. By inspection, find all constant solutions of the DE.

- (**b**) Solve the differential equation in part (a). A CAS may be useful for integration.
- (c) Use a graphing utility to obtain the graphs of all solutions that satisfy the initial condition W(0) = 2.
- **20. Evaporation** An outdoor decorative pond in the shape of a hemispherical tank is to be filled with water pumped into the tank through an inlet in its bottom. Suppose that the radius of the tank is R = 10 ft, that water is pumped in at a rate of π ft³/min, and that the tank is initially empty. See Figure (226, As the tank fills, it loses water through as portion. Assume that the rate of evaporation as population to the area A of the surface of the water and that the constant of proportionality is k = 0.01.

The area of change dV/dt of the volume of the water a time *t* is a net rate. Use this net rate to determine a differential equation for the height *h* of the water at time *t*. The volume of the water shown in the figure is $V = \pi Rh^2 - \frac{1}{3}\pi h^3$, where R = 10. Express the area of the surface of the water $A = \pi r^2$ in terms of *h*.

- (**b**) Solve the differential equation in part (a). Graph the solution.
- (c) If there were no evaporation, how long would it take the tank to fill
- (d) With evaporation, what is the depth of the water at the time found in part (c)? Will the tank ever be filled? Prove your assertion



FIGURE 3.2.6 Decorative pond in Problem 20

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21. Doomsday Equation Consider the differential equation

$$\frac{dP}{dt} = kP^{1+t}$$

where k > 0 and $c \ge 0$. In Section 3.1 we saw that in the case c = 0 the linear differential equation dP/dt = kP is a mathematical model of a population P(t) that exhibits unbounded growth over the infinit time interval $[0, \infty)$, that is, $P(t) \rightarrow \infty$ as $t \rightarrow \infty$. See Example 1 on page 84.

(a) Suppose for c = 0.01 that the nonlinear differential equation

$$\frac{dP}{dt} = kP^{1.01}, k > 0,$$

is a mathematical model for a population of small animals, where time t is measured in months. Solve the differential equation subject to the initial condition P(0) = 10 and the fact that the animal population has doubled in 5 months.

- (b) The differential equation in part (a) is called a **doomsday equation** because the population P(t)exhibits unbounded growth over a finite time interval (0, T), that is, there is some time T such $P(t) \rightarrow \infty$ as $t \rightarrow T^-$. Find T.
- (c) From part (a), what is P(50)? P(100)?
- 22. Doomsday or Extinction Suppose model (4) is modified to b

If a > 0, b > 10means of a phase portrait (see page 39) that, depending on the initial condition $P(0) = P_0$, the mathematical model could include a doomsday scenario $(P(t) \rightarrow \infty)$ or an extinction scenario $(P(t) \rightarrow 0)$.

(b) Solve the initial-value problem

$$\frac{dP}{dt} = P(0.0005P - 0.1), P(0) = 300.$$

Show that this model predicts a doomsday for the population in a finite time T.

(c) Solve the differential equation in part (b) subject to the initial condition P(0) = 100. Show that this model predicts extinction for the population as $t \rightarrow \infty$.

Project Problems

23. Regression Line Read the documentation for your CAS on scatter plots (or scatter diagrams) and leastsquares linear fit The straight line that best fits a set of

data points is called a regression line or a least squares line. Your task is to construct a logistic model for the population of the United States, defining f(P)in (2) as an equation of a regression line based on the population data in the table in Problem 4. One way of

doing this is to approximate the left-hand side $\frac{1}{P}\frac{dP}{dt}$ of

the first equation in (2), using the forward difference quotient in place of dP/dt:

$$Q(t) = \frac{1}{P(t)} \frac{P(t+h) - P(t)}{h}$$

(a) Make a table of the values t, P(t), and Q(t) using $t = 0, 10, 20, \dots, 160$ and h = 10. For example, the first line of the table should contain t = 0, P(0), and Q(0). With P(0) = 3.929 and P(10) = 5.308,

$$Q(0) = \frac{1}{P(0)} \frac{P(10) - P(0)}{10} = 0.035.$$

Note that Q(160) depends on the 1960 census population P(170). Look up this value.

(b) Use a CAS to obtain a scatter plot of the data (P(t), Q(t)) compared in part (a). Also use a CAS to find, an equation of the regression line and to superimpose its graph on the scatter plot.

? the optimized superimpose its graph on the scatter plot. the optimize is the equation of the regression line found in part (b). Solve the model in part (c) using the initial condi-tion P(t) = 2.022tion P(0) = 3.929.

- (e) Use a CAS to obtain another scatter plot, this time of the ordered pairs (t, P(t)) from your table in part (a). Use your CAS to superimpose the graph of the solution in part (d) on the scatter plot.
- (f) Look up the U.S. census data for 1970, 1980, and 1990. What population does the logistic model in part (c) predict for these years? What does the model predict for the U.S. population P(t) as $t \rightarrow \infty$?
- **24. Immigration Model** (a) In Examples 3 and 4 of Section 2.1 we saw that any solution P(t) of (4) possesses the asymptotic behavior $P(t) \rightarrow a/b$ as $t \rightarrow \infty$ for $P_0 > a/b$ and for $0 < P_0 < a/b$; as a consequence the equilibrium solution P = a/b is called an attractor. Use a root-finding application of a CAS (or a graphic calculator) to approximate the equilibrium solution of the immigration model

$$\frac{dP}{dt} = P(1 - P) + 0.3e^{-P}.$$

(b) Use a graphing utility to graph the function $F(P) = P(1 - P) + 0.3e^{-P}$. Explain how this graph can be used to determine whether the number found in part (a) is an attractor.

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(c) Use a numerical solver to compare the solution curves for the IVPs

$$\frac{dP}{dt} = P(1-P), \quad P(0) = P_0$$

for $P_0 = 0.2$ and $P_0 = 1.2$ with the solution curves for the IVPs

$$\frac{dP}{dt} = P(1-P) + 0.3e^{-P}, \quad P(0) = P_0$$

for $P_0 = 0.2$ and $P_0 = 1.2$. Superimpose all curves on the same coordinate axes but, if possible, use a different color for the curves of the second initial-value problem. Over a long period of time, what percentage increase does the immigration model predict in the population compared to the logistic model?

- **25.** What Goes Up... In Problem 16 let t_a be the time it takes the cannonball to attain its maximum height and let t_d be the time it takes the cannonball to fall from the maximum height to the ground. Compare the value of t_a with the value of t_d and compare the magnitude of the impact velocity v_i with the initial velocity v_0 . See Problem 48 in Exercises 3.1. A root-finding application of a CAS might be useful here. [*Hint*: Use the model in Problem 15 when the cannonball is falling.]
- **26. Skydiving** A skydiver is equipped with a stopwatch and an altimeter. As shown in Figure 3.2.7, he opposition parachute 25 seconds after exiting a plane nying as an altitude of 20,000 feet and observes that his altitude is 14,800 feet. Assume that his resultance is proportional to the square of the international velocity, his thittel velocity of a serving the plane is zero. In sec. 32 ft/s².
 - (a) Find the distance s(t), measured from the plane, the skydiver has traveled during freefall in time *t*. [*Hint*: The constant of proportionality *k* in the model given in Problem 15 is not specified. Use the expression for terminal velocity v_t obtained in part (b) of Problem 15 to eliminate *k* from the IVP. Then eventually solve for v_t .]
 - (b) How far does the skydiver fall and what is his velocity at t = 15 s?



FIGURE 3.2.7 Skydiver in Problem 26

27. Hitting Bottom A helicopter hovers 500 feet above a large open tank full of liquid (not water). A dense compact object weighing 160 pounds is dropped (released from rest) from the helicopter into the liquid. Assume that air resistance is proportional to instantaneous velocity v while the object is in the air and that viscous damping is proportional to v^2 after the object has entered the liquid. For air take $k = \frac{1}{4}$, and for the liquid take k = 0.1. Assume that the positive direction is downward. If the tank is 75 feet high, determine the time and the impact velocity when the object hits the bottom of the tank. [Hint: Think in terms of two distinct IVPs. If you use (13), be careful in removing the absolute value sign. You might compare the velocity when the object hits the liquid—the initial velocity for the second problem—with the terminal velocity v_t of the object falling through the liquid.]

28. Old Man River... In Figure 3.2.8(a) suppose that the y-axis and the dashed vertical line x = 1 represent, respectively, the straight west and east beaches of a river that is 1 mile wide. The river flows northward with a velocity \mathbf{v}_r , where $|\mathbf{v}_r| = v_r$ mi/h is a constant. A man enters the current at the point (1, 0) on the east shore and swims in a direction and rate relative to the river given by the vector \mathbf{v}_s , where the speed $\mathbf{v}_r = v_s \min/h$ is a constant. The man wants to rand the west beach exactly at (0, 0) and solve the substantial model are a manner that keeps his velocity vector \mathbf{v}_s and \mathbf{v}_s and \mathbf{v}_s and \mathbf{v}_s and \mathbf{v}_s and \mathbf{v}_s mi/h is a constant. The man wants to rand the west beach exactly at (0, 0) and solve the substantial model are a manner that keeps his velocity vector \mathbf{v}_s and $\mathbf{v$

$$=\frac{v_s y - v_r \sqrt{x^2 + y^2}}{v_s x}.$$

C

dy

dx

[*Hint*: The velocity **v** of the swimmer along the path or curve shown in Figure 3.2.8 is the resultant $\mathbf{v} = \mathbf{v}_s + \mathbf{v}_r$. Resolve \mathbf{v}_s and \mathbf{v}_r into components in the *x*- and



FIGURE 3.2.8 Path of swimmer in Problem 28

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When these new rates are decreased by rates proportional to the number of interactions, we obtain another nonlinear model:

$$\frac{dx}{dt} = a_1 x - b_1 x^2 - c_1 xy = x(a_1 - b_1 x - c_1 y)$$

$$\frac{dy}{dt} = a_2 y - b_2 y^2 - c_2 xy = y(a_2 - b_2 y - c_2 x),$$
(13)

where all coefficients are positive. The linear system (10) and the nonlinear systems (11) and (13) are, of course, called **competition models.**

Networks An electrical network having more than one loop also gives rise to simultaneous differential equations. As shown in Figure 3.3.3, the current $i_1(t)$ splits in the directions shown at point B_1 , called a *branch point* of the network. By **Kirchhoff's first la** we can write

$$i_1(t) = i_2(t) + i_3(t).$$
 (14)

We can also apply **Kirchhoff's second law** to each loop. For loop $A_1B_1B_2A_2A_1$, summing the voltage drops across each part of the loop gives

$$E(t) = i_1 R_1 + L_1 \frac{di_2}{dt} + i_2 R_2.$$
(15)

Similarly, for loop $A_1B_1C_1C_2B_2A_2A_1$ we fin

$$E(t) = i_1 R_1 + L_2 \frac{di_3}{dt}.$$
 (16)

Using (14) to eliminate i_1 in (15) and (16) yields two linear first-order equations for the currents $i_2(t)$ and $i_3(t)$:

$$L_{3} \frac{di_{3}}{di_{4}} - 189_{i_{1}i_{2}} + R_{1}i_{3} = E(t)$$

$$L_{3} \frac{di_{3}}{di_{4}} - 189_{i_{1}i_{2}} + R_{1}i_{3} = E(t).$$
(17)

We have a sum exercise (see Problem 14 in Exercises 3.3) to show that the system of differential equations describing the currents $i_1(t)$ and $i_2(t)$ in the network containing a resistor, an inductor, and a capacitor shown in Figure 3.3.4 is

$$L\frac{di_{1}}{dt} + Ri_{2} = E(t)$$

$$RC\frac{di_{2}}{dt} + i_{2} - i_{1} = 0.$$
(18)



FIGURE 3.3.3 Network whose model is given in (17)



EXERCISES 3.3

Radioactive Series

- 1. We have not discussed methods by which systems of first-order differential equations can be solved. Nevertheless, systems such as (2) can be solved with no knowledge other than how to solve a single linear first order equation. Find a solution of (2) subject to the initial conditions $x(0) = x_0$, y(0) = 0, z(0) = 0.
- 2. In Problem 1 suppose that time is measured in days, that the decay constants are $k_1 = -0.138629$ and $k_2 = -0.004951$, and that $x_0 = 20$. Use a graphing utility to obtain the graphs of the solutions x(t), y(t), and z(t)

Answers to selected odd-numbered problems begin on page ANS-4.

on the same set of coordinate axes. Use the graphs to approximate the half-lives of substances *X* and *Y*.

- **3.** Use the graphs in Problem 2 to approximate the times when the amounts *x*(*t*) and *y*(*t*) are the same, the times when the amounts *x*(*t*) and *z*(*t*) are the same, and the times when the amounts *y*(*t*) and *z*(*t*) are the same. Why does the time that is determined when the amounts *y*(*t*) and *z*(*t*) are the same.
- **4.** Construct a mathematical model for a radioactive series of four elements *W*, *X*, *Y*, and *Z*, where *Z* is a stable element.

Mixtures

- **5.** Consider two tanks *A* and *B*, with liquid being pumped in and out at the same rates, as described by the system of equations (3). What is the system of differential equations if, instead of pure water, a brine solution containing 2 pounds of salt per gallon is pumped into tank *A*?
- 6. Use the information given in Figure 3.3.5 to construct a mathematical model for the number of pounds of salt $x_1(t)$, $x_2(t)$, and $x_3(t)$ at time *t* in tanks *A*, *B*, and *C*, respectively.



FIGURE 3.3.5 Mixing tanks in Problem 6

7. Two very large tanks *A* and *B* are each partially filled with 100 gallons of brine. Initially, 100 pounds of salt is dissolved in the solution in tank *A* and 50 pounds of salt is dissolved in the solution in tank *B*. The system is closed in that the well-stirred liquid is purper to the between the tanks, as shown in Figure 1.36.



FIGURE 3.3.6 Mixing tanks in Problem 7

- (a) Use the information given in the figure to construct a mathematical model for the number of pounds of salt $x_1(t)$ and $x_2(t)$ at time t in tanks A and B, respectively.
- (b) Find a relationship between the variables $x_1(t)$ and $x_2(t)$ that holds at time *t*. Explain why this relationship makes intuitive sense. Use this relationship to help find the amount of salt in tank *B* at t = 30 min.
- 8. Three large tanks contain brine, as shown in Figure 3.3.7. Use the information in the figure to construct a mathematical model for the number of pounds of salt $x_1(t)$,

 $x_2(t)$, and $x_3(t)$ at time *t* in tanks *A*, *B*, and *C*, respectively. Without solving the system, predict limiting values of $x_1(t), x_2(t)$, and $x_3(t)$ as $t \rightarrow \infty$.



FIGURE 3.3.7 Mixing tanks in Problem 8

Predator-Prey Models

9. Consider the Lotka-Volterra predator-prey model defined b

$$\frac{dx}{dt} = -0.1x + 0.02xy$$
$$\frac{dy}{dt} = 0.2y - 0.025xy,$$

where the population x(t) (predators) and y(t) (prey) are measured in chousands. Suppose x(0) = 6 and x(t) = 0. Use a numerical solver to graph x(t) and y(t). Use the graphs to approximate the time t > 0 when the two populations are first equal. Use the graphs to approximate the period of each population.

Competition Models

10. Consider the competition model defined by

$$\frac{dx}{dt} = x(2 - 0.4x - 0.3y)$$
$$\frac{dy}{dt} = y(1 - 0.1y - 0.3x),$$

where the populations x(t) and y(t) are measured in thousands and t in years. Use a numerical solver to analyze the populations over a long period of time for each of the following cases:

(a)
$$x(0) = 1.5$$
, $y(0) = 3.5$
(b) $x(0) = 1$, $y(0) = 1$
(c) $x(0) = 2$, $y(0) = 7$
(d) $x(0) = 4.5$, $y(0) = 0.5$

11. Consider the competition model defined by

$$\frac{dx}{dt} = x(1 - 0.1x - 0.05y)$$
$$\frac{dy}{dt} = y(1.7 - 0.1y - 0.15x)$$

where the populations x(t) and y(t) are measured in thousands and t in years. Use a numerical solver to

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But this determinant is simply the Wronskian evaluated at x = t, and by assumption, $W \neq 0$. If we define $G(x) = C_1y_1(x) + C_2y_2(x)$, we observe that G(x) satisfies the differential equation since it is a superposition of two known solutions; G(x) satisfie the initial conditions

$$G(t) = C_1 y_1(t) + C_2 y_2(t) = k_1$$
 and $G'(t) = C_1 y'_1(t) + C_2 y'_2(t) = k_2$;

and Y(x) satisfies the *same* linear equation and the *same* initial conditions. Because the solution of this linear initial-value problem is unique (Theorem 4.1.1), we have Y(x) = G(x) or $Y(x) = C_1y_1(x) + C_2y_2(x)$.

EXAMPLE 7 General Solution of a Homogeneous DE

The functions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of the homogeneous linear equation y'' - 9y = 0 on the interval $(-\infty,)$. By inspection the solutions are linearly independent on the *x*-axis. This fact can be corroborated by observing that the Wronskian

$$W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$$

for every *x*. We conclude that y_1 and y_2 form a fundamental set of solutions, and consequently, $y = c_1 e^{3x} + c_2 e^{-3x}$ is the general solution of the equation on the interval.



EXAMPLE 9 General Solution of a Homogeneous DE

The functions $y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ satisfy the third-order equation y''' - 6y'' + 11y' - 6y = 0. Since

$$W(e^{x}, e^{2x}, e^{3x}) = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0$$

for every real value of x, the functions y_1 , y_2 , and y_3 form a fundamental set of solutions on $(-\infty, \)$. We conclude that $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$ is the general solution of the differential equation on the interval.

4.1.3 NONHOMOGENEOUS EQUATIONS

Any function y_p , free of arbitrary parameters, that satisfies (7) is said to be a **particular** solution or **particular integral** of the equation. For example, it is a straightforward task to show that the constant function $y_p = 3$ is a particular solution of the nonhomogeneous equation y'' + 9y = 27.

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Discussion Problems

4.2

37. Let $n = 1, 2, 3, \ldots$. Discuss how the observations $D^n x^{n-1} = 0$ and $D^n x^n = n!$ can be used to find the general solutions of the given differential equations. (a) y'' = 0**(b)** v''' = 0(a) $u^{(4)} = 0$

(a) $y = 0$	(b) $y = 0$	(c) $y = 0$
(d) $y'' = 2$	(e) $y''' = 6$	(f) $y^{(4)} = 24$

- **38.** Suppose that $y_1 = e^x$ and $y_2 = e^{-x}$ are two solutions of a homogeneous linear differential equation. Explain why $y_3 = \cosh x$ and $y_4 = \sinh x$ are also solutions of the equation.
- **39.** (a) Verify that $y_1 = x^3$ and $y_2 = |x|^3$ are linearly independent solutions of the differential equation $x^2y'' - 4xy' + 6y = 0$ on the interval $(-\infty,)$.
 - (b) Show that $W(y_1, y_2) = 0$ for every real number *x*. Does this result violate Theorem 4.1.3? Explain.
 - (c) Verify that $Y_1 = x^3$ and $Y_2 = x^2$ are also linearly independent solutions of the differential equation in part (a) on the interval $(-\infty,)$.
 - (d) Find a solution of the differential equation satisfying y(0) = 0, y'(0) = 0.

- (e) By the superposition principle, Theorem 4.1.2, both linear combinations $y = c_1y_1 + c_2y_2$ and $Y = c_1 Y_1 + c_2 Y_2$ are solutions of the differential equation. Discuss whether one, both, or neither of the linear combinations is a general solution of the differential equation on the interval $(-\infty,)$.
- **40.** Is the set of functions $f_1(x) = e^{x+2}$, $f_2(x) = e^{x-3}$ linearly dependent or linearly independent on $(-\infty,)$? Discuss.
- **41.** Suppose y_1, y_2, \ldots, y_k are k linearly independent solutions on $(-\infty, \)$ of a homogeneous linear *n*th-order differential equation with constant coefficients. By Theorem 4.1.2 it follows that $y_{k+1} = 0$ is also a solution of the differential equation. Is the set of solutions $y_1, y_2, \ldots, y_k, y_{k+1}$ linearly dependent or linearly independent on $(-\infty,)$? Discuss.
- **42.** Suppose that y_1, y_2, \ldots, y_k are k nontrivial solutions of a homogeneous linear *n*th-order differential equation with constant coefficients and that k = n + 1. Is the set of solutions y_1, y_2, \ldots, y_k linearly dependent or linearly independent on $(-\infty,)$? Discuss.

Preview Section 4. In the preceding section we saw that the general solution of a homogeneous TON. second-order differential equation

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a tion

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$
⁽¹⁾

is a linear combination $y = c_1y_1 + c_2y_2$, where y_1 and y_2 are solutions that constitute a linearly independent set on some interval I. Beginning in the next section, we examine a method for determining these solutions when the coefficients of the differential equation in (1) are constants. This method, which is a straightforward exercise in algebra, breaks down in a few cases and yields only a single solution y_1 of the DE. It turns out that we can construct a second solution y_2 of a homogeneous equation (1) (even when the coefficients in (1) are variable) provided that we know a nontrivial solution y_1 of the DE. The basic idea described in this section is that equation (1) can be reduced to a linear *first-o der DE by means of a substitution* involving the known solution y_1 . A second solution y_2 of (1) is apparent after this first-order di ferential equation is solved.

> **Example 1** Reduction of Order Suppose that y_1 denotes a nontrivial solution of (1) and that y_1 is defined on an interval I. We seek a second solution y_2 so that the set consisting of y_1 and y_2 is linearly independent on *I*. Recall from Section 4.1 that if y_1 and y_2 are linearly independent, then their quotient y_2/y_1 is nonconstant on *I*—that is, $y_2(x)/y_1(x) = u(x)$ or $y_2(x) = u(x)y_1(x)$. The function u(x) can be found by substituting $y_2(x) = u(x)y_1(x)$ into the given differential equation. This method is called **reduction** of order because we must solve a linear first-order di ferential equation to find u.

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By choosing $c_1 = 1$ and $c_2 = 0$, we find from $y = u(x)y_1(x)$ that a second solution of equation (3) is

$$y_2 = y_1(x) \int \frac{e^{-\int P(x) \, dx}}{y_1^2(x)} \, dx.$$
 (5)

It makes a good review of differentiation to verify that the function $y_2(x)$ defined in (5) satisfies equation (3) and that y_1 and y_2 are linearly independent on any interval on which $y_1(x)$ is not zero.

EXAMPLE 2 A Second Solution by Formula (5)

The function $y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$. Find the general solution of the differential equation on the interval (0,).

SOLUTION From the standard form of the equation,

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0,$$

$$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4} dx \quad \leftarrow e^{3\int dx/x} = e^{\ln x^3} = x^3$$

$$= x^2 \int \frac{dx}{x} = x^2 \ln x.$$

 $x^{-} \int \frac{dx}{x^{4}} dx \quad \leftarrow e^{3fdx/x} = e^{4nx^{4}} = x^{3}$ $= x^{2} \int \frac{dx}{x} = x^{2} \ln x.$ The general solution on the interval (0) Listeron by $y = c_{1}y_{1} + c_{2}y_{2}$; that is, $y = c_{1}x^{2} + c_{2}x^{2} \ln x.$ **NoteSale.**(0) Listeron by $y = c_{1}y_{1} + c_{2}y_{2}$; that is, $y = c_{1}x^{2} + c_{2}x^{2} \ln x.$ (1) The derivation of the origination of the provided of the matrix of the ma

the associated homogeneous equation is known. See Problems 17-20 in Exercises 4.2.

EXERCISES 4.2

Answers to selected odd-numbered problems begin on page ANS-5.

In Problems 1–16 the indicated function $y_1(x)$ is a solution of the given differential equation. Use reduction of order or formula (5), as instructed, to find a second solution $y_2(x)$.

- 1. $y'' 4y' + 4y = 0; y_1 = e^{2x}$
- **2.** $y'' + 2y' + y = 0; \quad y_1 = xe^{-x}$
- 3. $y'' + 16y = 0; y_1 = \cos 4x$
- 4. $y'' + 9y = 0; y_1 = \sin 3x$
- 5. $y'' y = 0; y_1 = \cosh x$
- 6. $y'' 25y = 0; \quad y_1 = e^{5x}$

7. $9y'' - 12y' + 4y = 0; \quad y_1 = e^{2x/3}$ 8. $6y'' + y' - y = 0; \quad y_1 = e^{x/3}$ **9.** $x^2y'' - 7xy' + 16y = 0; \quad y_1 = x^4$ **10.** $x^2y'' + 2xy' - 6y = 0; y_1 = x^2$ **11.** $xy'' + y' = 0; \quad y_1 = \ln x$ **12.** $4x^2y'' + y = 0; \quad y_1 = x^{1/2} \ln x$ **13.** $x^2y'' - xy' + 2y = 0; \quad y_1 = x\sin(\ln x)$ **14.** $x^2y'' - 3xy' + 5y = 0; \quad y_1 = x^2 \cos(\ln x)$



Solve
$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0.$$

SOLUTION The auxiliary equation $m^4 + 2m^2 + 1 = (m^2 + 1)^2 = 0$ has roots $m_1 = m_3 = i$ and $m_2 = m_4 = -i$. Thus from Case II the solution is

$$y = C_1 e^{ix} + C_2 e^{-ix} + C_3 x e^{ix} + C_4 x e^{-ix}.$$

By Euler's formula the grouping $C_1e^{ix} + C_2e^{-ix}$ can be rewritten as

$$c_1 \cos x + c_2 \sin x$$

after a relabeling of constants. Similarly, $x(C_3e^{ix} + C_4e^{-ix})$ can be expressed as $x(c_3 \cos x + c_4 \sin x)$. Hence the general solution is

$$y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x.$$

Example 4 illustrates a special case when the auxiliary equation has repeated complex roots. In general, if $m_1 = \alpha + i\beta$, $\beta > 0$ is a complex root of multiplicity k of an auxiliary equation with real coefficients, then its conjugate $m_2 = \alpha - i\beta$ is also a root of multiplicity k. From the 2k complex-valued solutions

$$e^{(\alpha+i\beta)x}$$
, $xe^{(\alpha+i\beta)x}$, $x^2e^{(\alpha+i\beta)x}$, ..., $x^{k-1}e^{(\alpha+i\beta)x}$,
 $e^{(\alpha-i\beta)x}$, $xe^{(\alpha-i\beta)x}$, $x^2e^{(\alpha-i\beta)x}$, ..., $x^{k-1}e^{(\alpha-i\beta)x}$,

we conclude, with the aid of Euler's formula, that the general solution of the corresponding differential equation must be contributed integration of the 2k real linearly independent solutions.

e
$$\cos \beta x$$
 $\sin \beta x$, $\sin \beta x$, $\sin \beta x$, \ldots , $x^{k-1}e^{\alpha x}\cos \beta x$,
e <sup>$\alpha x\sin \beta x$, $xe^{4}\sin \beta x$, $xe^{4}\sin \beta x$, \ldots , $x^{k-1}e^{\alpha x}\sin \beta x$.
In Example regionerity $k = 2$, $\alpha = 0$, and $\beta = 1$.
Of course the most difficult aspect of solving constant-coefficient differentiation of degree greater than two. For each $x = 0$, we must solve $3w^3 + 5w^2 + 10w^2$</sup>

In Example 5 (Jentify k = 2, $\alpha = 0$, and $\beta = 1$. Of course the most difficult aspect of solving constant-coefficient differential equations is finding roots of auxiliary equations of degree greater than two. For example, to solve 3y''' + 5y'' + 10y' - 4y = 0, we must solve $3m^3 + 5m^2 + 10m - 4 = 0$. Something we can try is to test the auxiliary equation for rational roots. Recall that if $m_1 = p/q$ is a rational root (expressed in lowest terms) of an auxiliary equation $a_n m^n + \cdots + a_1 m + a_0 = 0$ with integer coefficients, then *p* is a factor of a_0 and *q* is a factor of a_n . For our specific cubic auxiliary equation, all the factors of $a_0 = -4$ and $a_n = 3$ are $p: \pm 1, \pm 2, \pm 4$ and $q: \pm 1, \pm 3$, so the possible rational roots are $p/q: \pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$. Each of these numbers can then be tested—say, by synthetic division. In this way we discover both the root $m_1 = \frac{1}{3}$ and the factorization

$$3m^3 + 5m^2 + 10m - 4 = (m - \frac{1}{3})(3m^2 + 6m + 12).$$

The quadratic formula then yields the remaining roots $m_2 = -1 + \sqrt{3}i$ and $m_3 = -1 - \sqrt{3}i$. Therefore the general solution of 3y''' + 5y'' + 10y' - 4y = 0 is $y = c_1 e^{x/3} + e^{-x}(c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x)$.



Use of Computers Finding roots or approximation of roots of auxiliary equations is a routine problem with an appropriate calculator or computer software. Polynomial equations (in one variable) of degree less than five can be solved by means of algebraic formulas using the *solve* commands in *Mathematica* and *Maple*. For auxiliary equations of degree five or greater it might be necessary to resort to numerical commands such as **NSolve** and **FindRoot** in *Mathematica*. Because of their capability of solving polynomial equations, it is not surprising that these computer

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53. $y = c_1 \cos 3x + c_2 \sin 3x$ **54.** $y = c_1 \cosh 7x + c_2 \sinh 7x$

56. $y = c_1 + c_2 e^{2x} \cos 5x + c_3 e^{2x} \sin 5x$

57.
$$y = c_1 + c_2 x + c_3 e^8$$

58. $y = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x$



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44.

FIGURE 4.3.3 Graph for Problem 44

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When $\alpha = 0$ and n = 1, a special case of (7) is

$$(D^2 + \beta^2) \begin{cases} \cos \beta x \\ \sin \beta x \end{cases} = 0.$$
 (8)

For example, $D^2 + 16$ will annihilate any linear combination of sin 4x and cos 4x.

We are often interested in annihilating the sum of two or more functions. As we have just seen in Examples 1 and 2, if *L* is a linear differential operator such that $L(y_1) = 0$ and $L(y_2) = 0$, then *L* will annihilate the linear combination $c_1y_1(x) + c_2y_2(x)$. This is a direct consequence of Theorem 4.1.2. Let us now suppose that L_1 and L_2 are linear differential operators with constant coefficients such that L_1 annihilates $y_1(x)$ and L_2 annihilates $y_2(x)$, but $L_1(y_2) \neq 0$ and $L_2(y_1) \neq 0$. Then the *product* of differential operators L_1L_2 annihilates the sum $c_1y_1(x) + c_2y_2(x)$. We can easily demonstrate this, using linearity and the fact that $L_1L_2 = L_2L_1$:

$$L_{1}L_{2}(y_{1} + y_{2}) = L_{1}L_{2}(y_{1}) + L_{1}L_{2}(y_{2})$$

= $L_{2}L_{1}(y_{1}) + L_{1}L_{2}(y_{2})$
= $L_{2}[L_{1}(y_{1})] + L_{1}[L_{2}(y_{2})] = 0.$

For example, we know from (3) that D^2 annihilates 7 - x and from (8) that $D^2 + 16$ annihilates sin 4x. Therefore the product of operators $D^2(D^2 + 16)$ will annihilate the linear combination $7 - x + 6 \sin 4x$

Note The difference operator that annihilates a function is not unique. We saw integer C_{2} of Example 1 that D + 3 will annihilate e^{-3x} , but so will differential annihilate e^{-3x} . (Verify this.) and material of course, when we seek a differential annihilator for a function y = y(x), we want the operator of *lowest possible order* that does the job. **EVALUATE:** Undetermined Coefficient this brings us to the point of the preceding differential cussion. Suppose that L(x) = 1 for x = 1 for x = 1.

Condetermined Coefficient This brings us to the point of the preceding discussion. Suppose that L(y) = g(x) is a linear differential equation with constant coefficients and that the input g(x) consists of finite sums and products of the functions listed in (3), (5), and (7)—that is, g(x) is a linear combination of functions of the form

k (constant), x^m , $x^m e^{\alpha x}$, $x^m e^{\alpha x} \cos \beta x$, and $x^m e^{\alpha x} \sin \beta x$,

where *m* is a nonnegative integer and α and β are real numbers. We now know that such a function g(x) can be annihilated by a differential operator L_1 of lowest order, consisting of a product of the operators D^n , $(D - \alpha)^n$, and $(D^2 - 2\alpha D + \alpha^2 + \beta^2)^n$. Applying L_1 to both sides of the equation L(y) = g(x)yields $L_1L(y) = L_1(g(x)) = 0$. By solving the *homogeneous higher-order* equation $L_1L(y) = 0$, we can discover the *form* of a particular solution y_p for the original *nonhomogeneous* equation L(y) = g(x). We then substitute this assumed form into L(y) = g(x) to find an explicit particular solution. This procedure for determining y_p , called the **method of undetermined coefficients** is illustrated in the next several examples.

Before proceeding, recall that the general solution of a nonhomogeneous linear differential equation L(y) = g(x) is $y = y_c + y_p$, where y_c is the complementary function—that is, the general solution of the associated homogeneous equation L(y) = 0. The general solution of each equation L(y) = g(x) is defined on the interval $(-\infty,)$.

160 CHAPTER 4 HIGHER-ORDER DIFFERENTIAL EQUATIONS

Integrating
$$u'_1 = \frac{W_1}{W} = -\frac{1}{12}$$
 and $u'_2 = \frac{W_2}{W} = \frac{1}{12} \frac{\cos 3x}{\sin 3x}$

gives $u_1 = -\frac{1}{12}x$ and $u_2 = \frac{1}{36} \ln |\sin 3x|$. Thus a particular solution is

$$y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|.$$

The general solution of the equation is

$$y = y_c + y_p = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{12} x \cos 3x + \frac{1}{36} (\sin 3x) \ln|\sin 3x|.$$
(11)

Equation (11) represents the general solution of the differential equation on, say, the interval $(0, \pi/6)$.

EXAMPLE Constants of Integration When computing the indefinite integrals of u'_1 and u'_2 , we need not introduce any constants. This is because

$$y = y_c + y_p = c_1y_1 + c_2y_2 + (u_1 + a_1)y_1 + (u_2 + b_1)y_2$$

= $(c_1 + a_1)y_1 + (c_2 + b_1)y_2 + u_1y_1 + u_2y_2$
= $C_1y_1 + C_2y_2 + u_1y_1 + u_2y_2$.



Since the foregoing integrals are nonelementary, we are forced to write

$$y_p = \frac{1}{2} e^x \int_{x_0}^x \frac{e^{-t}}{t} dt - \frac{1}{2} e^{-x} \int_{x_0}^x \frac{e^t}{t} dt,$$

and so $y = y_c + y_p = c_1 e^x + c_2 e^{-x} + \frac{1}{2} e^x \int_{x_0}^x \frac{e^{-t}}{t} dt - \frac{1}{2} e^{-x} \int_{x_0}^x \frac{e^t}{t} dt.$ (12)

In Example 3 we can integrate on any interval $[x_0, x]$ that does not contain the origin. We will solve the equation in Example 3 by an alternative method in Section 4.8.

Higher-Order Equations The method that we have just examined for non homogeneous second-order differential equations can be generalized to linear nth-order equations that have been put into the standard form

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \dots + P_1(x)y' + P_0(x)y = f(x).$$
(13)

If $y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is the complementary function for (13), then a particular solution is

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x),$$

where the u'_k , k = 1, 2, ..., n are determined by the *n* equations

$$y_{1}u'_{1} + y_{2}u'_{2} + \cdots + y_{n}u'_{n} = 0$$

$$y'_{1}u'_{1} + y'_{2}u'_{2} + \cdots + y'_{n}u'_{n} = 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_{1}^{(n-1)}u'_{1} + y_{2}^{(n-1)}u'_{2} + \cdots + y_{n}^{(n-1)}u'_{n} = f(x).$$
(14)

The first n - 1 equations in this system, like $y_1u'_1 + y_2u'_2 = 0$ in (8), are assumptions that are made to simplify the resulting equation after $y_p = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x)$ is substituted in (13). In this case Cramer's Rule gives

$$u'_k=\frac{W_k}{W}, \quad k=1,2,\ldots,n,$$

where *W* is the Wronskian of y_1, y_2, \ldots, y_n and W_k is the determinant obtained by replacing the *k*th column of the Wronskian by the column consisting of the righthand side of (14)—that is, the column consisting of $(0, 0, \ldots, f(x))$. When n = 2, we get (9). When n = 3, the particular solution is $y_p = u_1y_1 + u_2y_2 + u_3y_3$, where y_1, y_2 , and y_3 constitute a linearly independent set of solutions of the associated homogeneous DE and u_1, u_2, u_3 are determined from

$$u'_1 = \frac{W_1}{W}, \qquad u'_2 = \frac{W_2}{W}, \qquad u'_3 = \frac{W_3}{W},$$
 (15)

$$W_{1} = \begin{vmatrix} 0 & y_{2} & y_{3} \\ 0 & y'_{2} & y'_{3} \\ f(x) & y''_{2} & y''_{3} \end{vmatrix}, \quad W_{2} = \begin{vmatrix} y_{1} & 0 & y_{3} \\ y'_{1} & 0 & y'_{3} \\ y''_{1} & f(x) & y''_{3} \end{vmatrix}, \quad W_{3} = \begin{vmatrix} y_{1} & y_{2} & y_{3} \\ y'_{2} & y'_{3} \\ y''_{2} & y''_{3} \end{vmatrix}, \quad and \quad W = \begin{vmatrix} y_{1} & y_{2} & y_{3} \\ y'_{1} & y'_{2} & y'_{3} \\ y''_{1} & y''_{2} & y''_{3} \end{vmatrix}$$

See Holdel's 20 in Exercises 4.6. **REMARKS** (i) Variation of parameters has a distinct advantage over the method of undetermined coefficients in that it will *always* yield a particular solution y_p provided that the associated homogeneous equation can be solved. The present method is not limited to a function f(x) that is a combination of the four types listed on page 140. As we shall see in the next section, variation of parameters, unlike undetermined coefficients, is applicable to linear DEs with variable coefficients

(*ii*) In the problems that follow, do not hesitate to simplify the form of y_p . Depending on how the antiderivatives of u'_1 and u'_2 are found, you might not obtain the same y_p as given in the answer section. For example, in Problem 3 in Exercises 4.6 both $y_p = \frac{1}{2} \sin x - \frac{1}{2} x \cos x$ and $y_p = \frac{1}{4} \sin x - \frac{1}{2} x \cos x$ are valid answers. In either case the general solution $y = y_c + y_p$ simplifies to $y = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x$. Why?

EXERCISES 4.6

Answers to selected odd-numbered problems begin on page ANS-6.

In Problems $1-18$ solve e tion of parameters.	ach differential equation by varia-	5. $y'' + y = \cos^2 x$	6. $y'' + y = \sec^2 x$
$1 y'' + y = \sec x$	2 $y'' + y = \tan x$	7. $y'' - y = \cosh x$	8. $y'' - y = \sinh 2x$
3. $y'' + y = \sin x$	4. $y'' + y = \sec \theta \tan \theta$	9. $y'' - 4y = \frac{e^{2x}}{x}$	10. $y'' - 9y = \frac{9x}{x^{3x}}$

11.
$$y'' + 3y' + 2y = \frac{1}{1 + e^x}$$

12. $y'' - 2y' + y = \frac{e^x}{1 + x^2}$
13. $y'' + 3y' + 2y = \sin e^x$
14. $y'' - 2y' + y = e^t \arctan t$
15. $y'' + 2y' + y = e^{-t} \ln t$
16. $2y'' + 2y' + y = 4\sqrt{x}$
17. $3y'' - 6y' + 6y = e^x \sec x$
18. $4y'' - 4y' + y = e^{x/2}\sqrt{1 - x^2}$

In Problems 19–22 solve each differential equation by variation of parameters, subject to the initial conditions y(0) = 1, y'(0) = 0.

19.
$$4y'' - y = xe^{x/2}$$

20. $2y'' + y' - y = x + 1$
21. $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$
22. $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$

In Problems 23 and 24 the indicated functions are known linearly independent solutions of the associated homogeneous differential equation on (0,). Find the general solution of the given nonhomogeneous equation.

23.
$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{3/2};$$

 $y_1 = x^{-1/2} \cos x, y_2 = x^{-1/2} \sin x$
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24.
$$x^2y'' + xy' + y = \sec(\ln x);$$

 $y_1 = \cos(\ln x), y_2 = \sin(\ln x)$

In Problems 25–28 solve the given third-order differential equation by variation of parameters.

25.
$$y''' + y' = \tan x$$

26. $y''' + 4y' = \sec 2x$
27. $y''' - 2y'' - y' + 2y = e^{4x}$
28. $y''' - 3y'' + 2y' = \frac{e^{2x}}{1 + e^x}$

Discussion Problems

In Problems 29 and 30 discuss how the methods of undetermined coefficients and variation of parameters can be combined to solve the given differential equation. Carry out your ideas.

29.
$$3y'' - 6y' + 30y = 15 \sin x + e^x \tan 3x$$

30. $y'' - 2y' + y = 4x^2 - 3 + x^{-1}e^x$

31. What are the intervals of definition of the general solutions in Problems 1, 7, 9, and 182 Discuss why the interval of definition of the general value of the motent (0, ...).

32. Einforthe general solution of $x^4y'' + x^3y' - 4x^2y = 1$ given that $y_1 = x^2$ is a solution of the associated homogeneous end tion.

- REVIEW MATERIAL
 - Review the concept of the auxiliary equation in Section 4.3.

INTRODUCTION The same relative ease with which we were able to find explicit solutions of higher-order linear differential equations with constant coefficients in the preceding sections does not, in general, carry over to linear equations with variable coefficients. We shall see in Chapter 6 that when a linear DE has variable coefficients, the best that we can *usually* expect is to find a solution in the form of an infinite series. However, the type of differential equation that we consider in this section is an exception to this rule; it is a linear equation with variable coefficients whose general solution can always be expressed in terms of powers of *x*, sines, cosines, and logarithmic functions. Moreover, its method of solution is quite similar to that for constant-coefficient equations in that an auxiliary equation must be solved.

Cauchy-Euler Equation A linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 x \frac{dy}{dx} + a_0 y = g(x),$$

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The next example illustrates the solution of a third-order Cauchy-Euler equation.

Solve
$$x^3 \frac{d^3 y}{dx^3} + 5x^2 \frac{d^2 y}{dx^2} + 7x \frac{dy}{dx} + 8y = 0.$$

SOLUTION The first three derivatives of $y = x^m$ are

$$\frac{dy}{dx} = mx^{m-1}, \qquad \frac{d^2y}{dx^2} = m(m-1)x^{m-2}, \qquad \frac{d^3y}{dx^3} = m(m-1)(m-2)x^{m-3},$$

so the given differential equation becomes

$$x^{3}\frac{d^{3}y}{dx^{3}} + 5x^{2}\frac{d^{2}y}{dx^{2}} + 7x\frac{dy}{dx} + 8y = x^{3}m(m-1)(m-2)x^{m-3} + 5x^{2}m(m-1)x^{m-2} + 7xmx^{m-1} + 8x^{m}$$
$$= x^{m}(m(m-1)(m-2) + 5m(m-1) + 7m + 8)$$
$$= x^{m}(m^{3} + 2m^{2} + 4m + 8) = x^{m}(m+2)(m^{2} + 4) = 0.$$

In this case we see that $y = x^m$ will be a solution of the differential equation for $m_1 = -2$, $m_2 = 2i$, and $m_3 = -2i$. Hence the general solution is $y = c_1 x^{-2} + c_2 \cos(2 \ln x) + c_3 \sin(2 \ln x)$.

Nonhomogeneous Equations The method of undetermined coefficient described in Sections 4.5 and 4.6 does not carry over, *ingeneral*, to nonhomogeneous linear differential equations with variable officients. Consequently, in our next example the method of variation of parameters is employed.

EXAMPLE Variation 5 13 ameters
Solve
$$x^2y'' + 3.6^+ + y = 2x^4e^x$$
.
Constrained to the equation is nonhomogeneous, we first solve the associated
progeneous equation. From the auxiliary equation $(m - 1)(m - 3) = 0$ we fin y_c
 $= c_1x + c_2x^3$. Now before using variation of parameters to find a particular solution
 $y_p = u_1y_1 + u_2y_2$, recall that the formulas $u'_1 = W_1/W$ and $u'_2 = W_2/W$, where W_1 ,
 W_2 , and W are the determinants defined on page 158, were derived under the assump-
tion that the differential equation has been put into the standard form $y'' + P(x)y' + y''$

Q(x)y = f(x). Therefore we divide the given equation by x^2 , and from

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x$$

we make the identification $f(x) = 2x^2e^x$. Now with $y_1 = x$, $y_2 = x^3$, and

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3, \quad W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x, \quad W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x,$$

we fin
$$u'_1 = -\frac{2x^3e^x}{2x^3} = -x^2e^x$$
 and $u'_2 = \frac{2x^3e^x}{2x^3} = e^x$

The integral of the last function is immediate, but in the case of u'_1 we integrate by parts twice. The results are $u_1 = -x^2e^x + 2xe^x - 2e^x$ and $u_2 = e^x$. Hence $y_p = u_1y_1 + u_2y_2$ is

$$y_p = (-x^2 e^x + 2x e^x - 2e^x)x + e^x x^3 = 2x^2 e^x - 2x e^x.$$

Finally,
$$y = y_c + y_p = c_1 x + c_2 x^3 + 2x^2 e^x - 2x e^x$$
.

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can be expressed as the superposition of two solutions:

$$y(x) = y_h(x) + y_p(x),$$
 (4)

where $y_h(x)$ is the solution of the associated homogeneous DE with nonhomogeneous initial conditions

$$y'' + P(x)y' + Q(x)y = 0, \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$
(5)

and $y_p(x)$ is the solution of the nonhomogeneous DE with homogeneous (that is, zero) initial conditions

$$y'' + P(x)y' + Q(x)y = f(x), \quad y(x_0) = 0, \quad y'(x_0) = 0.$$
 (6)

In the case where the coefficients P and Q are constants the solution of the IVP (5) presents no difficulties: We use the method of Section 4.3 to find the general solution of the homogeneous DE and then use the given initial conditions to determine the two constants in that solution. So we will focus on the solution of the IVP (6). Because of the zero initial conditions, the solution of (6) could describe a physical system that is initially at rest and so is sometimes called a **rest solution**.

Green's Function If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions on the interval *I* of the associated homogeneous form of (2), then a particular solution of the nonhomogeneous equation (2) on the interval *I* can be found by variation of parameters. Recall from (3) of Section 4.6, the form of this solution is

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$
(7)

The variable coefficients $u_1(x)$ and $u_2(x)$ in (7) are defined by (9) of Section 4.6

$$u_1'(x) = -\frac{y_2(x)f(x)}{VC}, \quad (2xO - \frac{y_1 \cdot y_2(x)}{W}.$$
(8)

The linear independence of f(x) and $y_2(x)$ on the interval I guarantees that the Wronskian $X = W_2(x)$, $y_2(x)$, $y_2(x) \neq 0$ from I x in I. If x and x_0 are numbers in I, then integrating the derivatives $u'_1(x)$ in $u'_2(x)$ in (8) on the interval $[x_0, x]$ and substituting the results into (7) gas.

Because of (2) the 2(A die constant
(1) be best of the integration of (2)
we can move these functions instant
the definite integrals
$$y_p(x) = y_1(x) \int_{x_0}^x \frac{-y_2(t)f(t)}{W(t)} dt + y_2(x) \int_{x_0}^x \frac{y_1(t)f(t)}{W(t)} dt$$

$$= \int_{x_0}^x \frac{-y_1(x)y_2(t)}{W(t)} f(t) dt + \int_{x_0}^x \frac{y_1(t)y_2(x)}{W(t)} f(t) dt,$$
(9)
where
$$W(t) = W(y_1(t), y_2(t)) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

From the properties of the definite integral, the two integrals in the second line of (9) can be rewritten as a single integral

$$y_p(x) = \int_{x_0}^x G(x, t) f(t) dt.$$
 (10)

The function G(x, t) in (10),

$$G(x, t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}$$
(11)

is called the Green's function for the differential equation (2).

Observe that a Green's function (11) depends only on the fundamental solutions $y_1(x)$ and $y_2(x)$ of the associated homogeneous differential equation for (2) and *not* on the forcing function f(x). Therefore all linear second-order differential equations (2) with the same left-hand side but with different forcing functions have the same the Green's function. So an alternative title for (11) is the **Green's function for the second-order differential operator** $L = D^2 + P(x)D + Q(x)$.

Here at least one of the numbers y_0 or y_1 is assumed to be nonzero. If both y_0 and y_1 are **0**, then the solution of the IVP is y = 0.

Important. Read this

paragraph a second time.

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4.8.2 **BOUNDARY-VALUE PROBLEMS**

In contrast to a second-order IVP, in which y(x) and y'(x) are specified at the same point, a BVP for a second-order DE involves conditions on y(x) and y'(x) that are specified at two different points x = a and x = b. Conditions such as

$$y(a) = 0$$
, $y(b) = 0$; $y(a) = 0$, $y'(b) = 0$; $y'(a) = 0$, $y'(b) = 0$.

are just special cases of the more general homogeneous boundary conditions:

$$A_1 y(a) + B_1 y'(a) = 0 (22)$$

$$A_2 y(b) + B_2 y'(b) = 0, (23)$$

where A_1, A_2, B_1 , and B_2 are constants. Specificall, our goal is to find a integral solution $y_p(x)$ that is analogous to (10) for nonhomogeneous boundary-value problems of the form

$$y'' + P(x)y' + Q(x)y = f(x),$$

$$A_1y(a) + B_1y'(a) = 0$$

$$A_2y(b) + B_2y'(b) = 0.$$
(24)

In addition to the usual assumptions that P(x), Q(x), and f(x) are continuous on [a, b], we assume that the homogeneous problem

$$y'' + P(x)y' + Q(x)y = 0,A_1y(a) + B_1y'(a) = 0A_2y(b) + B_2y'(b) = 0,$$

possesses only the trivial solution y = 0. This latter assumption is sufficient to guarantee that a unique solution of (24) exists and is given by an integral $y_p(x) = \int_a^b G(x, t) f(t) dt$, where G(x, t) is a Green's function.

) is again the variation of parame-The starting point in the construction of Gters formulas (7) and (8)

EXAMPLE 13 Function Subject $y_1(x)$ and $y_2(x)$ are linearly independent of the mon [a, b] of the associate (homogeneous form of the DE in (24) and that x is a number in the interval [a, b]. Unlike the construction of (9) where we started by integration the converties in (8) over the same interval, we now integrate the firs of biomer (s) on [b, x] and the second equation in (8) on [a, x]: $\int x_1 (x) f(x)$

$$u_1(x) = -\int_b^x \frac{y_2(t)f(t)}{W(t)}dt \quad \text{and} \quad u_2(x) = \int_a^x \frac{y_1(t)f(t)}{W(t)}dt.$$
 (25)

The reason for integrating $u'_1(x)$ and $u'_2(x)$ over different intervals will become clear shortly. From (25), a particular solution $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ of the DE is

here we used the minus
sign in (25) to reverse
the limits of integration

$$y_p(x) = y_1(x) \int_x^b \frac{y_2(t)f(t)}{W(t)} dt + y_2(x) \int_a^x \frac{y_1(t)f(t)}{W(t)} dt$$

$$y_p(x) = \int_a^x \frac{y_2(x)y_1(t)}{W(t)} f(t) dt + \int_x^b \frac{y_1(x)y_2(t)}{W(t)} f(t) dt.$$
(26)

or

The right-hand side of (26) can be written compactly as a single integral

$$y_p(x) = \int_a^b G(x, t) f(t) dt,$$
 (27)

where the function G(x, t) is

$$G(x,t) = \begin{cases} \frac{y_1(t)y_2(x)}{W(t)}, & a \le t \le x\\ \frac{y_1(x)y_2(t)}{W(t)}, & x \le t \le b. \end{cases}$$
(28)

Method of Solution Consider the simple system of linear first-orde equations

$$\frac{dx}{dt} = 3y$$

or, equivalently,
$$\begin{aligned} Dx - 3y &= 0\\ 2x - Dy &= 0. \end{aligned}$$
(1)

Operating on the first equation in (1) by D while multiplying the second by -3 and then adding eliminates y from the system and gives $D^2x - 6x = 0$. Since the roots of the auxiliary equation of the last DE are $m_1 = \sqrt{6}$ and $m_2 = -\sqrt{6}$, we obtain

$$x(t) = c_1 e^{-\sqrt{6}t} + c_2 e^{\sqrt{6}t}.$$
 (2)

Multiplying the first equation in (1) by 2 while operating on the second by D and then subtracting gives the differential equation for y, $D^2y - 6y = 0$. It follows immediately that

$$y(t) = c_3 e^{-\sqrt{6}t} + c_4 e^{\sqrt{6}t}.$$
(3)

Now (2) and (3) do not satisfy the system (1) for every choice of c_1 , c_2 , c_3 , and c_4 because the system itself puts a constraint on the number of parameters in a solution that can be chosen arbitrarily. To see this, observe that substituting x(t) and y(t)into the first equation of the original system (1) gives, after simplificatio

$$(-\sqrt{6}c_1 - 3c_3)e^{-\sqrt{6}t} + (\sqrt{6}c_1 - 3c_4)e^{\sqrt{6}t} = 0.$$

Since the latter expression is to be the all values of t, we must have $-\sqrt{6}c = 3c = 0$ and $\sqrt{6}c$ The end of 0. These two equations enable us to write c_3

nd
$$c_4 = \frac{\sqrt{6}}{3}c_2.$$
 (4)

$$(t) = c_1 e^{-\sqrt{6}t} + c_2 e^{\sqrt{6}t}, \qquad y(t) = -\frac{\sqrt{6}}{3}c_1 e^{-\sqrt{6}t} + \frac{\sqrt{6}}{3}c_2 e^{\sqrt{6}t}.$$

You are urged to substitute (2) and (3) into the second equation of (1) and verify that the same relationship (4) holds between the constants.

EXAMPLE 1 Solution by Elimination

Solve

$$Dx + (D + 2)y = 0$$

(D - 3)x - 2y = 0. (5)

SOLUTION Operating on the first equation by D - 3 and on the second by D and then subtracting eliminates x from the system. It follows that the differential equation for y is

$$[(D-3)(D+2)+2D]y = 0$$
 or $(D^2+D-6)y = 0.$

Since the characteristic equation of this last differential equation is $m^2 + m - 6 = (m - 2)(m + 3) = 0$, we obtain the solution

$$y(t) = c_1 e^{2t} + c_2 e^{-3t}.$$
 (6)

Eliminating y in a similar manner yields $(D^2 + D - 6)x = 0$, from which we fin

$$x(t) = c_3 e^{2t} + c_4 e^{-3t}.$$
(7)

where $A = \sqrt{c_1^2 + c_2^2}$. In this case the radian measured angle ϕ is defined in slightly different manner than in (7):

$$\begin{array}{ccc}
\sin &= \frac{c_2}{A} \\
\cos &= \frac{c_1}{A}
\end{array}$$

$$\tan &= \frac{c_2}{c_1}.$$
(7')

For example, in Example 2 with $c_1 = \frac{2}{3}$ and $c_2 = -\frac{1}{6}$, (7') indicates that $\tan \phi = -\frac{1}{4}$. Because $\sin \phi < 0$ and $\cos \phi > 0$ the angle ϕ lies in the fourth quadrant and so rounded to three decimal places $\phi = \tan^{-1}(-\frac{1}{4}) = -0.245$ rad. From (6') we obtain a second alternative form of solution (5):

$$x(t) = \frac{\sqrt{17}}{6}\cos(8t - (-0.245))$$
 or $x(t) = \frac{\sqrt{17}}{6}\cos(8t + 0.245).$

Graphical Interpretation Figure 5.1.4(a) illustrates the mass in Example 2 going through approximately two complete cycles of motion. Reading left to right, the first five positions (marked with black dots) correspond to the initial position of the mass below the equilibrium position $(x = \frac{2}{3})$, the mass passing through the equilibrium position for the first time heading upward (x = 0), the mass at its extreme displacement above the equilibrium position $(x = -\sqrt{17}/6)$, the mass at the equilibrium position for the second time heading downward (x = 0), and the mass at its extreme displacement below the equilibrium position $(x = \sqrt{17}/6)$. The black dots on the graph of (9), given in Figure 5.1.4(b), also agree with the five positionsiust given. Note, however, that in Figure 5.1.4(b) the positive direction in the tabove is the usual upward



FIGURE 5.1.4 Simple harmonic motion

The mass is initially released from rest $\frac{1}{2}$ unit (foot or meter) below the equilibrium position. The motion is damped ($\beta = 1.2$) and is being driven by an external periodic ($T = \pi/2$ s) force beginning at t = 0. Intuitively, we would expect that even with damping, the system would remain in motion until such time as the forcing function was "turned off," in which case the amplitudes would diminish. However, as the problem is given, $f(t) = 5 \cos 4t$ will remain "on" forever.

We first multiply the di ferential equation in (26) by 5 and solve

$$\frac{dx^2}{dt^2} + 6\frac{dx}{dt} + 10x = 0$$

by the usual methods. Because $m_1 = -3 + i$, $m_2 = -3 - i$, it follows that $x_c(t) = e^{-3t}(c_1 \cos t + c_2 \sin t)$. Using the method of undetermined coefficients, we assume a particular solution of the form $x_p(t) = A \cos 4t + B \sin 4t$. Differentiating $x_p(t)$ and substituting into the DE gives

$$x_p'' + 6x_p' + 10x_p = (-6A + 24B)\cos 4t + (-24A - 6B)\sin 4t = 25\cos 4t.$$

The resulting system of equations

$$-6A + 24B = 25, \quad -24A - 6B = 0$$

yields $A = -\frac{25}{102}$ and $B = \frac{50}{51}$. It follows that

$$x(t) = e^{-3t}(c_1 \cos t + c_2 \sin t) - \frac{25}{102} \cos 4t + \frac{50}{51} \sin 4t.$$
 (27)

When we set t = 0 in the above equation, we obtain $c_1 = \frac{38}{51}$. By differentiating the expression and then setting t = 0, we also find that $c_2 = -\frac{86}{51}$. Therefore the equation of motion is

$$x(t) = -\frac{3}{51} \left(\frac{3}{51} \cos t - \frac{35}{51} \sin t \right) - \frac{25}{102} \cos 4t + \frac{50}{51} \sin 4t. \quad (28) \equiv$$

Examplent and Steady-thte Terms When *F* is a periodic function, such as $F(t) = F_0 \sin x_t + F(t) = F_0 \cos \gamma t$, the general solution of (25) for $\lambda > 0$ is the sum of chorperiodic function $x_c(t)$ and a periodic function $x_p(t)$. Moreover, $x_c(t)$ dies off is time increases—that is, $\lim_{t\to\infty} x_c(t) = 0$. Thus for large values of time, the displacements of the mass are closely approximated by the particular solution $x_p(t)$. The complementary function $x_c(t)$ is said to be a **transient term** or **transient solution**, and the function $x_p(t)$, the part of the solution that remains after an interval of time, is called a **steady-state term** or **steady-state solution**. Note therefore that the effect of the initial conditions on a spring/mass system driven by *F* is transient. In the particular solution (28), $e^{-3t}(\frac{38}{51}\cos t - \frac{86}{51}\sin t)$ is a transient term, and $x_p(t) = -\frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t$ is a steady-state term. The graphs of these two terms and the solution (28) are given in Figures 5.1.12(a) and 5.1.12(b), respectively.

EXAMPLE 7 Transient/Steady-State Solutions

The solution of the initial-value problem

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 4\cos t + 2\sin t, \quad x(0) = 0, \quad x'(0) = x_1,$$

where x_1 is constant, is given by

$$x(t) = (x_1 - 2) e^{-t} \sin t + 2 \sin t.$$

Solution curves for selected values of the initial velocity x_1 are shown in Figure 5.1.13. The graphs show that the influence of the transient term is negligible for about $t > 3\pi/2$.







(b)





FIGURE 5.1.13 Graph of solution in Example 7 for various initial velocities x_1

- **35.** A mass *m* is attached to the end of a spring whose constant is *k*. After the mass reaches equilibrium, its support begins to oscillate vertically about a horizontal line *L* according to a formula h(t). The value of *h* represents the distance in feet measured from *L*. See Figure 5.1.21.
 - (a) Determine the differential equation of motion if the entire system moves through a medium offering a damping force that is numerically equal to $\beta(dx/dt)$.
 - (b) Solve the differential equation in part (a) if the spring is stretched 4 feet by a mass weighing 16 pounds and $\beta = 2$, $h(t) = 5 \cos t$, x(0) = x'(0) = 0.



FIGURE 5.1.21 Oscillating support in Problem 35

- **36.** A mass of 100 grams is attached to a spring whose constant is 1600 dynes/cm. After the mass reaches equilibrium, its support oscillates according to the formula $h(t) = \sin 8t$, where *h* represents displacement from its original position. See Problem 35 and Figure 5.1.21
 - (a) In the absence of damping, determine the equation of motion if the mass starts firm react from the equilibrium position
 - (c) At what times does the mass attain its extreme displacements?

s the mass pass

- (d) What are the maximum and minimum displacements?
- (e) Graph the equation of motion.

(b)

In Problems 37 and 38 solve the given initial-value problem.

37.
$$\frac{d^2x}{dt^2} + 4x = -5\sin 2t + 3\cos 2t$$
,
 $x(0) = -1$, $x'(0) = 1$
38. $\frac{d^2x}{dt^2} + 9x = 5\sin 3t$, $x(0) = 2$, $x'(0) = 0$

39. (a) Show that the solution of the initial-value problem

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \cos \gamma t, \quad x(0) = 0, \quad x'(0) = 0$$

is
$$x(t) = \frac{F_0}{\omega^2 - \gamma^2} (\cos \gamma t - \cos \omega t).$$

(b) Evaluate
$$\lim_{\gamma \to \omega} \frac{F_0}{\omega^2 - \gamma^2} (\cos \gamma t - \cos \omega t)$$
.

- **40.** Compare the result obtained in part (b) of Problem 39 with the solution obtained using variation of parameters when the external force is $F_0 \cos \omega t$.
- **41.** (a) Show that *x*(*t*) given in part (a) of Problem 39 can be written in the form

$$x(t) = \frac{-2F_0}{\omega^2 - \gamma^2} \sin \frac{1}{2} (\gamma - \omega) t \sin \frac{1}{2} (\gamma + \omega) t.$$

(b) If we define $\varepsilon = \frac{1}{2}(\gamma - \omega)$, show that when ε is small an approximate solution is

$$x(t) = \frac{F_0}{2\varepsilon\gamma}\sin\varepsilon t\sin\gamma t.$$

When ε is small, the frequency $\gamma/2\pi$ of the impressed force is close to the frequency $\omega/2\pi$ of free vibrations. When this occurs, the motion is as indicated in Figure 5.1.22. Oscillations of this kind are called **beats** and are due to the fact that the frequency of sin εt is quite small in comparison to the frequency of sin εt . The dashed curves, or envelope of the gappa $(\varepsilon x(t), \text{ are obtained from the graphs of } \pm F_0/2\varepsilon\gamma) \sin \varepsilon t$. Use a graphing this with various values of F_0 , ε , and γ to verify the graph in Figure 5.1.22.



FIGURE 5.1.22 Beats phenomenon in Problem 41

Computer Lab Assignments

42. Can there be beats when a damping force is added to the model in part (a) of Problem 39? Defend your position with graphs obtained either from the explicit solution of the problem

$$\frac{d^2x}{dt^2} + 2 \frac{dx}{dt} + \omega^2 x = F_0 \cos \gamma t, \quad x(0) = 0, \quad x'(0) = 0$$

or from solution curves obtained using a numerical solver.

43. (a) Show that the general solution of

l

$$\frac{d^2x}{dt^2} + 2 \quad \frac{dx}{dt} + \omega^2 x = F_0 \sin \gamma t$$

is one model of a free spring/mass system in which the damping force is proportional to the square of the velocity. One can then envision other kinds of models: linear damping and nonlinear restoring force, nonlinear damping and nonlinear restoring force, and so on. The point is that nonlinear characteristics of a physical system lead to a mathematical model that is nonlinear.

Notice in (2) that both $F(x) = kx^3$ and $F(x) = kx + k_1x^3$ are odd functions of x. To see why a polynomial function containing only odd powers of x provides a reasonable model for the restoring force, let us express F as a power series centered at the equilibrium position x = 0:

$$F(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

When the displacements x are small, the values of x^n are negligible for n sufficiently large. If we truncate the power series with, say, the fourth term, then $F(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$. For the force at x > 0,

$$F(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3,$$

and for the force at -x < 0,

$$F(-x) = c_0 - c_1 x + c_2 x^2 - c_3 x^3$$

to have the same magnitude but act in the opposite direction, we must have F(-x) = -F(x). Because this means that *F* is an odd function, we must have $c_0 = 0$ and $c_2 = 0$, and so $F(x) = c_1x + c_3x^3$. Had we used only the first two terms in the series, the same argument yields the linear function $F(x) = c_1x$. A restoring force with mixed powers, such as $F(x) = c_1x + c_2x^2$, and the corresponding vibrations are said to be **unsymmetrical**. In the next discussion we shall write $c_1 = k$ and $c_3 = k_1$.

Hard and constraints Let us take a closer look at the equation in (1) in the case i) which the rector in force is given by $F(x) = kx + k_1x^3$, k > 0. The spin is said to be hard f(x) > 0 and **soft** if $k_1 < 0$. Graphs of three types of restoring force are illustrated in Figure 5.3.1. The next example illustrates index two special cases of the differential equation $md^2x/dt^2 + kx + k_1x^3 = 0$, k > 0.

EXAMPLE 1 Comparison of Hard and Soft Springs

 d^2

The differential equations

$$\frac{d^2x}{dt^2} + x + x^3 = 0 (4)$$

$$\frac{x}{2} + x - x^3 = 0 \tag{5}$$

are special cases of the second equation in (2) and are models of a hard spring and a soft spring, respectively. Figure 5.3.2(a) shows two solutions of (4) and Figure 5.3.2(b) shows two solutions of (5) obtained from a numerical solver. The curves shown in red are solutions that satisfy the initial conditions x(0) = 2, x'(0) = -3; the two curves in blue are solutions that satisfy x(0) = 2, x'(0) = 0. These solution curves certainly suggest that the motion of a mass on the hard spring is oscillatory, whereas motion of a mass on the soft spring appears to be nonoscillatory. But we must be careful about drawing conclusions based on a couple of numerical solution curves. A more complete picture of the nature of the solutions of both of these equations can be obtained from the qualitative analysis discussed in Chapter 10.



FIGURE 5.3.1 Hard and soft springs









 $\theta(0) = \frac{1}{2}$, $\theta'(0) = 2$. The blue curve represents a periodic solution—the pendulum oscillating back and forth as shown in Figure 5.3.4(b) with an apparent amplitude $A \leq 1$. The red curve shows that θ increases without bound as time increases—the pendulum, starting from the same initial displacement, is given an initial velocity of magnitude great enough to send it over the top; in other words, the pendulum is whirling about its pivot as shown in Figure 5.3.4(c). In the absence of damping, the motion in each case is continued indefinitel . \equiv

Telephone Wires The first-order differential equation $dy/dx = W/T_1$ is equation (16) of Section 1.3. This differential equation, established with the aid of Figure 1.3.8 on page 26, serves as a mathematical model for the shape of a flex ible cable suspended between two vertical supports when the cable is carrying a vertical load. In Section 2.2 we solved this simple DE under the assumption that the vertical load carried by the cables of a suspension bridge was the weight of a horizontal roadbed distributed evenly along the x-axis. With $W = \rho x$, ρ the weight per unit length of the roadbed, the shape of each cable between the vertical supports turned out to be parabolic. We are now in a position to determine the shape of a uniform flexible cable hanging only under its own weight, such as a wire strung between two telephone posts. The vertical load is now the wire itself, and so if ρ is the linear density of the wire (measured, say, in pounds per feet) and s is the length of the segment P_1P_2 in Figure 1.3.8 then $W = \rho s$. Hence

$$\frac{dy}{dx} = \frac{\rho s}{T_1}.$$
(8)

(9)

$$\frac{dy}{dx} = \frac{p^2}{T_1}.$$
(8)
Since the arc length between points P_1 and P_2 is seen by

$$\mathbf{from} \quad \mathbf{from} \quad \mathbf{from}$$

Differentiating (8) with respect to x and using (10) lead to the second-order equation

$$\frac{d^2 y}{dx^2} = \frac{\rho}{T_1} \frac{ds}{dx} \qquad \text{or} \qquad \frac{d^2 y}{dx^2} = \frac{\rho}{T_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$
 (11)

In the example that follows we solve (11) and show that the curve assumed by the suspended cable is a catenary. Before proceeding, observe that the nonlinear second-order differential equation (11) is one of those equations having the form F(x, y', y'') = 0 discussed in Section 4.10. Recall that we have a chance of solving an equation of this type by reducing the order of the equation by means of the substitution u = y'.

EXAMPLE 3 A Solution of (11)

From the position of the y-axis in Figure 1.3.8 it is apparent that initial conditions associated with the second differential equation in (11) are y(0) = a and y'(0) = 0.

If we substitute u = y', then the equation in (11) becomes $\frac{du}{dx} = \frac{\rho}{T_1}\sqrt{1+u^2}$. Sepa-

rating variables, we find that

$$\int \frac{du}{\sqrt{1+u^2}} = \frac{\rho}{T_1} \int dx \quad \text{gives} \quad \sinh^{-1}u = \frac{\rho}{T_1}x + c_1.$$



FIGURE 5.3.6 Chain pulled upward by a constant force in Example 4

EXAMPLE 4 **Chain Pulled Upward by a Constant Force**

A uniform 10-foot-long chain is coiled loosely on the ground. One end of the chain is pulled vertically upward by means of constant force of 5 pounds. The chain weighs 1 pound per foot. Determine the height of the end above ground level at time t. See Figure 5.3.6.

SOLUTION Let us suppose that x = x(t) denotes the height of the end of the chain in the air at time t, v = dx/dt, and the positive direction is upward. For the portion of the chain that is in the air at time t we have the following variable quantities:

weight:
$$W = (x \text{ ft}) \cdot (1 \text{ lb/ft}) = x$$
,
mass: $m = W/g = x/32$,
net force: $F = 5 - W = 5 - x$.

Thus from (14) we have

$$\int_{1}^{\text{Product Rule}} dt \left(\frac{x}{32}v\right) = 5 - x \quad \text{or} \quad x \frac{dv}{dt} + v \frac{dx}{dt} = 160 - 32x.$$
(15)

Because v = dx/dt, the last equation becomes

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$$\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 32.7 = 160.$$
 (16)

The nonlinear second-order differential equation (16) has the form F(x, x', x'') = 0, which is the second of the two forms considered in Section 4.10 that can possibly be solved by r dicerch of order. To solve (16), we revert back to (15) and use v = x'second o souved bur recomb along winn the Chain can berowriter a Page $\int \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$ the second equation in (15) uO

$$v \frac{dv}{dx} + v^2 = 160 - 32x.$$
(17)

On inspection (17) might appear intractable, since it cannot be characterized as any of the first-order equations that were solved in Chapter 2. However, by rewriting (17) in differential form M(x, v)dx + N(x, v)dv = 0, we observe that although the equation

$$(v^2 + 32x - 160)dx + xv \, dv = 0 \tag{18}$$

is not exact, it can be transformed into an exact equation by multiplying it by an integrating factor. From $(M_v - N_x)/N = 1/x$ we see from (13) of Section 2.4 that an integrating factor is $e^{\int dx/x} = e^{\ln x} = x$. When (18) is multiplied by $\mu(x) = x$, the resulting equation is exact (verify). By identifying $\partial f / \partial x = xv^2 + 32x^2 - 160x$, $\partial f / \partial v = x^2 v$ and then proceeding as in Section 2.4, we obtain

$$\frac{1}{2}x^2v^2 + \frac{32}{3}x^3 - 80x^2 = c_1.$$
⁽¹⁹⁾

Since we have assumed that all of the chain is on the floor initially, we have x(0) = 0. This last condition applied to (19) yields $c_1 = 0$. By solving the algebraic equation $\frac{1}{2}x^2v^2 + \frac{32}{3}x^3 - 80x^2 = 0$ for v = dx/dt > 0, we get another first-orde differential equation,

$$\frac{dx}{dt} = \sqrt{160 - \frac{64}{3}x}.$$

- 14. A mass weighing 32 pounds stretches a spring 6 inches. The mass moves through a medium offering a damping force that is numerically equal to β times the instantaneous velocity. Determine the values of $\beta > 0$ for which the spring/mass system will exhibit oscillatory motion.
- 15. A spring with constant k = 2 is suspended in a liquid that offers a damping force numerically equal to 4 times the instantaneous velocity. If a mass *m* is suspended from the spring, determine the values of *m* for which the subsequent free motion is nonoscillatory.
- 16. The vertical motion of a mass attached to a spring is described by the IVP $\frac{1}{4}x'' + x' + x = 0$, x(0) = 4, x'(0) = 2. Determine the maximum vertical displacement of the mass.
- 17. A mass weighing 4 pounds stretches a spring 18 inches. A periodic force equal to $f(t) = \cos \gamma t + \sin \gamma t$ is impressed on the system starting at t = 0. In the absence of a damping force, for what value of γ will the system be in a state of pure resonance?
- **18.** Find a particular solution for $x'' + 2\lambda x' + \omega^2 x = A$, where A is a constant force.
- 19. A mass weighing 4 pounds is suspended from a spring whose constant is 3 lb/ft. The entire system is immersed in a fluid offering a damping force numerically equal to the instantaneous velocity. Beginning at t = 0, an external force equal to $f(t) = e^{-t}$ is impressed on the system. Determine the equation of motion if the mass initially released from rest at a point 2 for the but the equilibrium position.
- **20.** (a) Two springs are at ached in series a Galow 1 in Figure 2.76. If the mass of each pring edgnored, subw that the effective energy constant k of the system is defined to $1/x_0 = 1/k_1 + 1/k_2$.
 - (b) A mass weighing W pounds stretches a spring $\frac{1}{2}$ foot and stretches a different spring $\frac{1}{4}$ foot. The two springs are attached, and the mass is then attached to the double spring as shown in Figure 5.R.1. Assume that the motion is free and that there is no damping force present. Determine the equation of motion if the mass is initially released at a point 1 foot below the equilibrium position with a downward velocity of $\frac{2}{3}$ ft/s.
 - (c) Show that the maximum speed of the mass is $\frac{2}{3}\sqrt{3g+1}$.



FIGURE 5.R.1 Attached springs in Problem 20

- **21.** A series circuit contains an inductance of L = 1 h, a capacitance of $C = 10^{-4}$ f, and an electromotive force of E(t) = 100 sin 50t V. Initially, the charge q and current *i* are zero.
 - (a) Determine the charge q(t).
 - (b) Determine the current i(t).
 - (c) Find the times for which the charge on the capacitor is zero.
- 22. (a) Show that the current i(t) in an *LRC*-series circuit satisfies $L \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{1}{C}i = E'(t)$, where E'(t) denotes the derivative of E(t).
 - (b) Two initial conditions i(0) and i'(0) can be specifie for the DE in part (a). If i(0) = i₀ and q(0) = q₀, what is i'(0)?
- **23.** Consider the boundary-value problem

y'' + y = 0, $y(0) = y(2\pi)$, $y'(0) = y'(2\pi)$.

Show that except for the case $\lambda = 0$, there are two independent eigenfunctions corresponding to each eigenvalue.

24. A bead is construined to slide along a frictionless rod of length C. (b) not is rotating in a vertical plane with a ω is tant angular velocity ω about a pivot P fixed at the 3 midpoint of the rod, but the design of the pivot allows ticebead to move along the entire length of the rod. Let 0 denote the position of the bead relative to this rotating coordinate system as shown in Figure 5.R.2. To apply Newton's second law of motion to this rotating frame of reference, it is necessary to use the fact that the net force acting on the bead is the sum of the real forces (in this case, the force due to gravity) and the inertial forces (coriolis, transverse, and centrifugal). The mathematics is a little complicated, so we just give the resulting differential equation for *r*:

$$m\frac{d^2r}{dt^2} = m\omega^2 r - mg\sin\omega t.$$

- (a) Solve the foregoing DE subject to the initial conditions $r(0) = r_0$, $r'(0) = v_0$.
- (b) Determine the initial conditions for which the bead exhibits simple harmonic motion. What is the minimum length *L* of the rod for which it can accommodate simple harmonic motion of the bead?
- (c) For initial conditions other than those obtained in part (b), the bead must eventually fly off the rod. Explain using the solution r(t) in part (a).
- (d) Suppose $\omega = 1 \text{ rad/s}$. Use a graphing utility to graph the solution r(t) for the initial conditions $r(0) = 0, r'(0) = v_0$, where v_0 is 0, 10, 15, 16, 16.1, and 17.

Example Case III: If r_1 and r_2 are equal, then there always exist two linearly independent solutions of equation (1) of the form

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \qquad c_0 \neq 0,$$
 (21)

$$y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^{n+r_1}.$$
 (22)

Example 7 Finding a Second Solution When the difference $r_1 - r_2$ is a positive integer (Case II), we may or may not be able to find two solutions having the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$. This is something that we do not know in advance but is determined after we have found the indicial roots and have carefully examined the recurrence relation that defines the coefficients c_n . We just may be lucky enough to find two solutions that involve only powers of x, that is, $y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}$ (equation (19)) and $y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2}$ (equation (20) with C = 0). See Problem 31 in Exercises 6.3. On the other hand, in Example 4 we see that the difference of the indicial roots is a positive integer $(r_1 - r_2 = 1)$ and the method of Frobenius failed to give a second series solution. In this situation equation (20), with $C \neq 0$, indicates what the second solution looks like. Finally, when the difference $r_1 - r_2$ is a zero (Case III), the method of Frobenius fails to give a second series solution; the second solution (22) always contains a logarithm and can be shown to be equivalent to (20) with C = 1. One way to obtain the second solution with the logarithmic term is to use the fact that

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$$
(23)

is also a solution of y'' + P(x)y' + Q(x)y = 0 whenever y(x) is a known solution. We illustrate how to use (23) in the next example.

First Augeneral solution of xy solution of the known page **Revisited Using a CAS** the known solution given in Example 4, $y_1(x) = x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{144}x^4 + \cdots,$

we can construct a second solution $y_2(x)$ using formula (23). Those with the time, energy, and patience can carry out the drudgery of squaring a series, long division, and integration of the quotient by hand. But all these operations can be done with relative ease with the help of a CAS. We give the results:

$$y_{2}(x) = y_{1}(x) \int \frac{e^{-\int 0dx}}{[y_{1}(x)]^{2}} dx = y_{1}(x) \int \frac{dx}{\left[x - \frac{1}{2}x^{2} + \frac{1}{12}x^{3} - \frac{1}{144}x^{4} + \cdots\right]^{2}}$$

$$= y_{1}(x) \int \frac{dx}{\left[x^{2} - x^{3} + \frac{5}{12}x^{4} - \frac{7}{72}x^{5} + \cdots\right]} \quad \leftarrow \text{ after squaring}$$

$$= y_{1}(x) \int \left[\frac{1}{x^{2}} + \frac{1}{x} + \frac{7}{12} + \frac{19}{72}x + \cdots\right] dx \quad \leftarrow \text{ after long division}$$

$$= y_{1}(x) \left[-\frac{1}{x} + \ln x + \frac{7}{12}x + \frac{19}{144}x^{2} + \cdots\right] \quad \leftarrow \text{ after integrating}$$

$$= y_{1}(x) \ln x + y_{1}(x) \left[-\frac{1}{x} + \frac{7}{12}x + \frac{19}{144}x^{2} + \cdots\right],$$

 \equiv

 $y_2(x) = y_1(x) \ln x + \left[-1 - \frac{1}{2}x + \frac{1}{2}x^2 + \cdots \right].$ or ← after multiplying out

On the interval $(0, \infty)$ the general solution is $y = C_1 y_1(x) + C_2 y_2(x)$.

Note that the final form of y_2 in Example 5 matches (20) with C = 1; the series in the brackets corresponds to the summation in (20) with $r_2 = 0$.

REMARKS

(*i*) The three different forms of a linear second-order differential equation in (1), (2), and (3) were used to discuss various theoretical concepts. But on a practical level, when it comes to actually solving a differential equation using the method of Frobenius, it is advisable to work with the form of the DE given in (1).

(ii) When the difference of indicial roots $r_1 - r_2$ is a positive integer $(r_1 > r_2)$, it sometimes pays to iterate the recurrence relation using the smaller root r_2 first. See Problems 31 and 32 in Exercises 6.3

(iii) Because an indicial root r is a solution of a quadratic equation, it could be complex. We shall not, however, investigate this case.

(*iv*) If x = 0 is an irregular singular point, then we might not be able to fin any solution of the DE of form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$.

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In Problems 1–10 determine the singular ports of the liven differential equation. Classify each in that has not or irregular.

1.
$$y^{2}y^{2} + 4x^{2}y^{2} + 3y = 0$$

2. $x(x+3)^{2}y'' - y$ by a generative of the formula o

$$x \quad (x-1)^{3}$$

5.
$$(x^3 + 4x)y'' - 2xy' + 6y = 0$$

6.
$$x^2(x-5)^2y'' + 4xy' + (x^2-25)y = 0$$

7.
$$(x^2 + x - 6)y'' + (x + 3)y' + (x - 2)y = 0$$

8.
$$x(x^2 + 1)^2 y'' + y = 0$$

C

9.
$$x^{3}(x^{2}-25)(x-2)^{2}y'' + 3x(x-2)y' + 7(x+5)y = 0$$

10.
$$(x^3 - 2x^2 + 3x)^2 y'' + x(x - 3)^2 y' - (x + 1)y = 0$$

In Problems 11 and 12 put the given differential equation into form (3) for each regular singular point of the equation. Identify the functions p(x) and q(x).

11. $(x^2 - 1)y'' + 5(x + 1)y' + (x^2 - x)y = 0$ **12.** $xy'' + (x + 3)y' + 7x^2y = 0$

It or b ms 13 and 14, x = 0 is a regular singular point of the **Dial Efferential equation.** Use the general form of the indicial equation in (14) to find the indicial roots of the singularity. Without solving, discuss the number of series solutions you would expect to find using the method of Frobenius

ted odd-numbered problems begin on page ANS-10.

13.
$$x^2y'' + (\frac{5}{3}x + x^2)y' - \frac{1}{3}y = 0$$

14. $xy'' + y' + 10y = 0$

In Problems 15–24, x = 0 is a regular singular point of the given differential equation. Show that the indicial roots of the singularity do not differ by an integer. Use the method of Frobenius to obtain two linearly independent series solutions about x = 0. Form the general solution on $(0, \infty)$.

15.
$$2xy'' - y' + 2y = 0$$

16. $2xy'' + 5y' + xy = 0$
17. $4xy'' + \frac{1}{2}y' + y = 0$
18. $2x^2y'' - xy' + (x^2 + 1)y = 0$
19. $3xy'' + (2 - x)y' - y = 0$
20. $x^2y'' - (x - \frac{2}{9})y = 0$
21. $2xy'' - (3 + 2x)y' + y = 0$

Yet another equation, important because many DEs fit into its form by appropriate choices of the parameters, is

$$y'' + \frac{1 - 2a}{x}y' + \left(b^2c^2x^{2c-2} + \frac{a^2 - p^2c^2}{x^2}\right)y = 0, \qquad p \quad 0.$$
(18)

Although we shall not supply the details, the general solution of (18),

$$y = x^{a} \left[c_{1} J_{p}(bx^{c}) + c_{2} Y_{p}(bx^{c}) \right],$$
 (19)

can be found by means of a change in both the independent and the dependent variables: $z = bx^c$, $y(x) = \left(\frac{z}{b}\right)^{a/c} w(z)$. If p is not an integer, then Y_p in (19) can be replaced by J_{-n} .

EXAMPLE 3 **Using** (18)

Find the general solution of xy'' + 3y' + 9y = 0 on $(0, \infty)$.

SOLUTION By writing the given DE as

$$y'' + \frac{3}{x}y' + \frac{9}{x}y = 0,$$

we can make the following identifications with (18)

1 - 2a = 3, $b^2c^2 = 9$, 2c - 2 = -12 and $a^2 - p^2c^2 = 0$. The first and third equations implicitly r = -1 and $c = \frac{1}{2}$. With these values the second and fourth equation are solution by taking b = 6 and p = 2. From (19) we find that the control solution of the given DE on the interval $(0, \infty)$ is

EXAMPLE 4 The Aging Spring Revisited

review from the second Recall that in Section 5.1 we saw that one mathematical model for the free undamped motion of a mass on an aging spring is given by $mx'' + ke^{-\alpha t}x = 0$, $\alpha > 0$. We are now in a position to find the general solution of the equation. It is left as a problem

to show that the change of variables $s = \frac{2}{\alpha} \sqrt{\frac{k}{m}} e^{-\alpha t/2}$ transforms the differential equation of the aging spring into

$$s^2 \frac{d^2x}{ds^2} + s \frac{dx}{ds} + s^2 x = 0.$$

The last equation is recognized as (1) with $\nu = 0$ and where the symbols x and s play the roles of y and x, respectively. The general solution of the new equation is $x = c_1 J_0(s) + c_2 Y_0(s)$. If we resubstitute s, then the general solution of $mx'' + ke^{-\alpha t}x = 0$ is seen to be

$$x(t) = c_1 J_0 \left(\frac{2}{\alpha} \sqrt{\frac{k}{m}} e^{-\alpha t/2}\right) + c_2 Y_0 \left(\frac{2}{\alpha} \sqrt{\frac{k}{m}} e^{-\alpha t/2}\right).$$

See Problems 33 and 39 in Exercises 6.4.

The other model that was discussed in Section 5.1 of a spring whose characteristics change with time was mx'' + ktx = 0. By dividing through by m, we see that the equation $x'' + \frac{k}{m}tx = 0$ is Airy's equation $y'' + \alpha^2 xy = 0$. See Example 5 in Section 6.2. The general solution of Airy's differential equation can also be written in terms of Bessel functions. See Problems 34, 35, and 40 in Exercises 6.4.

In view of the property $\Gamma(1 + \alpha) = \alpha \Gamma(\alpha)$ and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ the values of $\Gamma(1 + \frac{1}{2} + n)$ for n = 0, n = 1, n = 2, and n = 3 are, respectively,

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(1 + \frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(1 + \frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2^2}\sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(1 + \frac{5}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5 \cdot 3}{2^3}\sqrt{\pi} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2^3 4 \cdot 2}\sqrt{\pi} = \frac{5!}{2^5 2!}\sqrt{\pi}$$

$$\Gamma\left(\frac{9}{2}\right) = \Gamma\left(1 + \frac{7}{2}\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right) = \frac{7 \cdot 5}{2^6 \cdot 2!}\sqrt{\pi} = \frac{7 \cdot 6 \cdot 5!}{2^6 \cdot 6 \cdot 2!}\sqrt{\pi} = \frac{7!}{2^7 3!}\sqrt{\pi}.$$

In general,

$$\Gamma(1 + \frac{1}{2} + n) = \frac{(2n+1)!}{2^{2n+1}n!} \sqrt{\pi}.$$

Hence
$$J_{1/2}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \frac{(2n+1)!}{2^{2n+1}n!} \sqrt{\pi}} \left(\frac{x}{2}\right)^{2n+1/2} = \sqrt{\frac{2}{\pi x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

From (2) of Section 6.1 you should recognize that the infinite series in the last line is the Maclaurin series for $\sin x$, and so we have shown that



FIGURE 6.4.5 Bessel functions of order $\frac{1}{2}$ (blue) and order $-\frac{1}{2}$ (red)

 $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$ (23) We leave it as an exercise to show that

$$0^{1/2}(x) \equiv \sqrt{\frac{2}{\pi x}} \cos x. \tag{24}$$

Set Figure 6.4.5 and Problems 4.2.2, and 38 in Exercises 6.4. If *n* is ariticler, ther $\nu = n + \frac{1}{2}$ is half an odd integer. Because $\cos(n + \frac{1}{2}) \pi = 0$ and $\sin(n + \frac{1}{2}) \pi = \cos n \pi = (-1)^n$, we see from (10) that $Y_{n+1/2}(x) = (-1)^{n+1} J_{-(n+1/2)}(x)$. For n = 0 and n = -1 we have, in turn, $Y_{1/2}(x) = -J_{-1/2}(x)$ and $Y_{-1/2}(x) = J_{1/2}(x)$. In view of (23) and (24) these results are the same as

$$Y_{1/2}(x) = -\sqrt{\frac{2}{\pi x} \cos x}$$
 (25)

$$Y_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$$
 (26)

and

Spherical Bessel Functions Bessel functions of half-integral order are used to define two more important functions:

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x)$$
 and $y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x).$ (27)

The function $j_n(x)$ is called the **spherical Bessel function of the first kin** and $y_n(x)$ is the **spherical Bessel function of the second kind.** For example, for n = 0 the expressions in (27) become

$$j_0(x) = \sqrt{\frac{\pi}{2x}} J_{1/2}(x) = \sqrt{\frac{\pi}{2x}} \sqrt{\frac{2}{\pi x}} \sin x = \frac{\sin x}{x}$$
$$y_0(x) = \sqrt{\frac{\pi}{2x}} Y_{1/2}(x) = -\sqrt{\frac{\pi}{2x}} \sqrt{\frac{2}{\pi x}} \cos x = -\frac{\cos x}{x}$$

and

end embedded in the ground, will deflect, or bend away, from the vertical under the influence of its own weight when its length or height exceeds a certain critical value. It can be shown that the angular deflection $\theta(x)$ of the column from the vertical at a point P(x) is a solution of the boundary-value problem:

$$EI\frac{d^2\theta}{dx^2} + \delta g(L-x)\theta = 0, \quad \theta(0) = 0, \quad \theta'(L) = 0,$$

where E is Young's modulus, I is the cross-sectional moment of inertia, δ is the constant linear density, and x is the distance along the column measured from its base. See Figure 6.4.7. The column will bend only for those values of L for which the boundary-value problem has a nontrivial solution.

- (a) Restate the boundary-value problem by making the change of variables t = L - x. Then use the results of a problem earlier in this exercise set to express the general solution of the differential equation in terms of Bessel functions.
- (b) Use the general solution found in part (a) to find a solution of the BVP and an equation which define the critical length L, that is, the smallest value of L for which the column will start to bend.



FIGURE 6.4.7 Beam in Problem 41

42. Buckling of a Thin Vertical Column In Example 4 of Section 5.2 we saw that when a constant vertical compressive force, or load, P was applied to a thin column of uniform cross section and hinged at both ends, the deflection y(x) is a solution of the BVP:

$$EI\frac{d^2y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0.$$

(a) If the bending stiffness factor EI is proportional to x, then EI(x) = kx, where k is a constant of proportionality. If EI(L) = kL = M is the maximum stiffness factor, then k = M L and so EI(x) = Mx L.

Use the information in Problem 37 to find a solution of

$$M\frac{x}{L}\frac{d^2y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0$$

if it is known that $\sqrt{x} Y_1(2\sqrt{\lambda x})$ is *not* zero at x = 0.

- (b) Use Table 6.4.1 to find the Euler load P_1 for the column.
- (c) Use a CAS to graph the first buckling mode $y_1(x)$ corresponding to the Euler load P_1 . For simplicity assume that $c_1 = 1$ and L = 1.
- 43. Pendulum of Varying Length For the simple pendulum described on page 220 of Section 5.3, suppose that the rod holding the mass *m* at one end is replaced by a flexible wire or string and that the wire is strung over a pulley at the point of support O in Figure 5.3.3. In this manner, while it is in motion in a vertical plane, the mass *m* can be raised or lowered. In other words, the length l(t) of the pendulum varies with time. Under the same assumptions leading to equation (6) in Section 5.3, it can be shown* that the differential equation for the displacement angle θ is now

$$l\theta'' + 2l'\theta' + g\sin\theta = 0.$$

If *l* increases at constant rate *v* and if $l(0) = l$

$$(l_0 + vt)\theta'' + 2v\theta' + g\theta = 0.$$
(34)

$$\frac{d^2\theta}{dx^2} + \frac{2}{x}\frac{d\theta}{dx} + \frac{g}{vx}\theta = 0.$$

- of equation (34) in terms of Bessel functions.
- (d) Use the general solution obtained in part (c) to solve the initial-value problem consisting of equation (34) and the initial conditions $\theta(0) = \theta_0$, $\theta'(0) = 0$. [Hints: To simplify calculations, use a further

change of variable $u = \frac{2}{v} \sqrt{g(l_0 + vt)} = 2 \sqrt{\frac{g}{v}} x^{1/2}$ Also, recall that (20) holds for both $J_1(u)$ and $Y_1(u)$.

$$J_1(u)Y_2(u) - J_2(u)Y_1(u) = -\frac{2}{\pi u}$$

will be helpful.]

(problem continues on page 270)

^{*}See Mathematical Methods in Physical Sciences, Mary Boas, John Wiley & Sons, Inc., 1966. Also see the article by Borelli, Coleman, and Hobson in Mathematics Magazine, vol. 58, no. 2, March 1985.

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(b)
$$\mathscr{L}\lbrace e^{5t}\rbrace = \int_0^\infty e^{5t} e^{-st} dt = \int_0^\infty e^{-(s-5)t} dt$$
$$= \frac{-e^{-(s-5)t}}{s-5} \Big|_0^\infty$$
$$= \frac{1}{s-5}.$$

In contrast to part (a), this result is valid for s = 5 because $\lim_{t \to \infty} e^{-(s-5)t} = 0$ demands s - 5 > 0 or s = 5.

EXAMPLE 4 Applying Definition 7.1.

Evaluate $\mathscr{L}{\sin 2t}$.

SOLUTION From Definition 7.1.1 and two applications of integration by parts we obtain

$$\mathcal{L}\{\sin 2t\} = \int_{0}^{\infty} e^{-st} \sin 2t \, dt = \frac{-e^{-st} \sin 2t}{s} \Big|_{0}^{\infty} + \frac{2}{s} \int_{0}^{\infty} e^{-st} \cos 2t \, dt$$

$$= \frac{2}{s} \int_{0}^{\infty} e^{-st} \cos 2t \, dt, \qquad s = 0$$

$$\lim_{t \to \infty} e^{-st} \cos 2t = 0, s = 0$$

$$= \frac{2}{s} \Big[\frac{-e^{-st} \cos 2t}{s} \Big|_{0}^{\infty} - \frac{2}{s} \int_{0}^{\infty} e^{-st} \sin t | t^{t} | t^{t} \Big]$$

$$= \frac{2}{s} \Big[\frac{4}{s} e^{-st} \cos 2t \Big|_{0}^{\infty} - \frac{2}{s} \int_{0}^{\infty} e^{-st} \sin t | t^{t} | t^{t} \Big]$$
At this point we have an equation over $\mathcal{L}\{\sin 2t\}$ on both sides of the equality.
Solving for that quantity velds at result

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^{2} + 4}, \qquad s = 0.$$

$$= \frac{2}{s} [s a \text{ Linear Transform} \text{ For a linear combination of functions we can write}$$

$$\int_{0}^{\infty} e^{-st} [\alpha f(t) + \beta g(t)] \, dt = \alpha \int_{0}^{\infty} e^{-st} f(t) \, dt + \beta \int_{0}^{\infty} e^{-st} g(t) \, dt$$

.

$$\int_0^\infty e^{-st} [\alpha f(t) + \beta g(t)] dt = \alpha \int_0^\infty e^{-st} f(t) dt + \beta \int_0^\infty e^{-st} g(t) dt$$

whenever both integrals converge for s = c. Hence it follows that

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathscr{L}\{f(t)\} + \beta \mathscr{L}\{g(t)\} = \alpha F(s) + \beta G(s).$$
(3)

Because of the property given in (3), \mathcal{L} is said to be a **linear transform.**

EXAMPLE 5 Linearity of the Laplace Transform

In this example we use the results of the preceding examples to illustrate the linearity of the Laplace transform.

(a) From Examples 1 and 2 we have for s = 0,

$$\mathscr{L}\left\{1+5t\right\} = \mathscr{L}\left\{1\right\} + 5\mathscr{L}\left\{t\right\} = \frac{1}{s} + \frac{5}{s^2}.$$

(b) From Examples 3 and 4 we have for s = 5,

$$\mathscr{L}\left\{4e^{5t} - 10\sin 2t\right\} = 4\mathscr{L}\left\{e^{5t}\right\} - 10\mathscr{L}\left\{\sin 2t\right\} = \frac{4}{s-5} - \frac{20}{s^2+4}.$$

EXERCISES 7.1

Answers to selected odd-numbered problems begin on page ANS-11.

In Problems 1–18 use Definition 7.1.1 to find $\mathscr{L}{f(t)}$. 2 $\mathbf{1.}\ f(t) = \begin{cases} -1, & 0 \le t < 1\\ 1, & t \ge 1 \end{cases}$ 2 2 **2.** $f(t) = \begin{cases} 4, & 0 \le t < 2\\ 0, & t \ge 2 \end{cases}$ 2 3 **3.** $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$ 3 3 $\mathbf{4.}\ f(t) = \begin{cases} 2t+1, & 0 \le t < 1\\ 0, & t \ge 1 \end{cases}$ **5.** $f(t) = \begin{cases} \sin t, & 0 \le t < \\ 0, & t \ge \end{cases}$ metric identity. **37.** *f* **6.** $f(t) = \begin{cases} 0, & 0 \le t < /2 \\ \cos t, & t \ge /2 \end{cases}$ **39.** $f(t) = \sin(4t + 5)$ **41.** We have encountered the **gamma function** $\Gamma(\alpha)$ in our 7. f(t)study of Bessel functions in Section 6.4 (page 258). One definition of this function is given by the improper integral $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt \quad \alpha = 0.$ Use this definition (a) how that $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$. 2, 2) 42. Use ration (2) how that $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$. 43. Use ration 41 and a change of variables to obtain the constalization $\Gamma(\alpha + 1)$ FIGURE 7.1.6 Graph for Problem 7 8. $\frac{489}{s^{\alpha+1}} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \alpha > -1,$ of the result in Theorem 7.1.1(b). In Problems 43-46 use Problems 41 and 42 and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ to find the Laplace transform of the given function. **43.** $f(t) = t^{-1/2}$ **45.** $f(t) = t^{3/2}$ FIGURE 7.1.8 Graph for Problem 9 **Discussion Problems** 10. f(t)47. Make up a function F(t) that is of exponential order but where f(t) = F'(t) is not of exponential order. Make up a function f that is not of exponential order but whose Laplace transform exists. FIGURE 7.1.9 Graph for Problem 10 **48.** Suppose that $\mathscr{L}{f_1(t)} = F_1(s)$ for $s = c_1$ and that $\mathscr{L}{f_2(t)} = F_2(s)$ for $s = c_2$. When does **11.** $f(t) = e^{t+7}$ **12.** $f(t) = e^{-2t-5}$ $\mathscr{L}\{f_1(t) + f_2(t)\} = F_1(s) + F_2(s)?$ 14. $f(t) = t^2 e^{-2t}$ **13.** $f(t) = te^{4t}$ 49. Figure 7.1.4 suggests, but does not prove, that the func-**15.** $f(t) = e^{-t} \sin t$ **16.** $f(t) = e^t \cos t$ tion $f(t) = e^{t^2}$ is not of exponential order. How does the observation that $t^2 = \ln M + ct$, for M = 0 and t **17.** $f(t) = t \cos t$ **18.** $f(t) = t \sin t$ sufficiently la ge, show that e^{t^2} Me^{ct} for any c? In Problems 19–36 use Theorem 7.1.1 to find $\mathcal{L}{f(t)}$. 50. Use part (c) of Theorem 7.1.1 to show that **20.** $f(t) = t^5$ **19.** $f(t) = 2t^4$ $\mathscr{L}\left\{e^{(a+ib)t}\right\} = \frac{s-a+ib}{(s-a)^2+b^2}$, where a and b are real **22.** f(t) = 7t + 3**21.** f(t) = 4t - 10

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23.
$$f(t) = t^2 + 6t - 3$$
24. $f(t) = -4t^2 + 16t + 9$ **25.** $f(t) = (t + 1)^3$ **26.** $f(t) = (2t - 1)^3$ **27.** $f(t) = 1 + e^{4t}$ **28.** $f(t) = t^2 - e^{-9t} + 5$ **29.** $f(t) = (1 + e^{2t})^2$ **30.** $f(t) = (e^t - e^{-t})^2$ **31.** $f(t) = 4t^2 - 5\sin 3t$ **32.** $f(t) = \cos 5t + \sin 2t$ **33.** $f(t) = \sinh kt$ **34.** $f(t) = \cosh kt$ **35.** $f(t) = e^t \sinh t$ **36.** $f(t) = e^{-t} \cosh t$

In Problems 37–40 find $\mathscr{L}{f(t)}$ by first using a trigono-

44. $f(t) = t^{1/2}$

46. $f(t) = 2t^{1/2} + 8t^{5/2}$

$$f(t) = \sin 2t \cos 2t \qquad 38. \quad f(t) = \cos^2 t$$

$$f(t) = \sin(4t + 5) \qquad 40. \quad f(t) = 10 \cos\left(t - \frac{1}{6}\right)$$

$$t) = \sin 2t \cos 2t$$

$$) = \sin 2i \cos 2i$$

$$= \sin 2t \cos 2t$$

$$t) = \sin 2t \cos 2t$$

$$t) = \sin 2t \cos 2t$$

$$t = \sin 2t \cos 2t$$

$$) = \sin 2t \cos 2t$$

$$= \sin 2t \cos 2t$$

$$f) = \sin 2t \cos 2t$$

$$t) = \sin 2t \cos 2t$$

$$) = \sin 2t \cos 2t$$
 3

$$) = \sin 2t \cos 2t$$

$$f(t) = \sin 2t \cos 2t \qquad 3$$

$$f) = \sin 2t \cos 2t$$
 3

$$t) = \sin 2t \cos 2t \qquad 3$$

$$t = \sin 2t \cos 2t$$

$$t) = \sin 2t \cos 2t$$

$$t) = \sin 2t \cos 2t$$

$$= \sin 2t \cos 2t$$
 3

$$f(t) = \sin 2t \cos 2t$$

$$\hat{c}(t) = \sin 2t \cos 2t$$

$$f(t) = \sin 2t \cos 2t$$

$$(t) = \sin 2t \cos 2t$$

$$f) = \sin 2t \cos 2t$$

$$f(t) = \sin 2t \cos 2t \qquad 3$$

$$(t) = \sin 2t \cos 2t$$

$$(t) = \sin 2t \cos 2t$$

$$(t) = \sin 2t \cos 2t$$

$$t$$
) = sin 2t cos 2t 3

$$f(t) = \sin 2t \cos 2t$$

$$f(t) = \sin 2t \cos 2t \qquad 3$$

$$f(t) = \sin 2t \cos 2t$$

$$(t) = \sin 2t \cos 2t$$

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$$t) = \sin 2t \cos 2t$$

$$f(t) = \sin 2t \cos 2t$$

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$$t = \sin 2t \cos 2t$$

$$t = \sin 2t \cos 2t$$

$$f(t) = \sin 2t \cos 2t$$

$$t) = \sin 2t \cos 2t$$

$$f(t) = \sin 2t \cos 2t$$

$$t) = \sin 2t \cos 2t$$

$$f(t) = \sin 2t \cos 2t$$

$$j = \sin 2i \cos 2i$$

$$-\sin 2i\cos 2i$$

$$-\sin 2i\cos 2i$$

$$-\sin 2i\cos 2i$$

$$-\sin 2i\cos 2i$$

$$= \sin 2t \cos 2t$$

$$) = \sin 2t \cos 2t$$

$$= \sin 2t \cos 2t$$

$$= \sin 2t \cos 2t$$

$$f) = \sin 2t \cos 2t$$

and $i^2 = -1$. Show how Euler's formula (page 133) can then be used to deduce the results

$$\mathcal{L}\left\{e^{at}\cos bt\right\} = \frac{s-a}{(s-a)^2 + b^2}$$
$$\mathcal{L}\left\{e^{at}\sin bt\right\} = \frac{b}{(s-a)^2 + b^2}.$$

- **51.** Under what conditions is a linear function $f(x) = mx + b, m \neq 0$, a linear transform?
- **52.** Explain why the function

$$f(t) = \begin{cases} t, & 0 \le t < 2\\ 4, & 2 < t < 5\\ 1/(t-5), & t & 5 \end{cases}$$

is not piecewise continuous on $[0, \infty)$.

53. Show that the function $f(t) = 1/t^2$ does not possess a Laplace transform. [*Hint*: Write $\mathscr{L} \{1/t^2\}$ as two improper integrals:

$$\mathscr{L}\left\{1/t^{2}\right\} = \int_{0}^{1} \frac{e^{-st}}{t^{2}} dt + \int_{1}^{\infty} \frac{e^{-st}}{t^{2}} dt = I_{1} + I_{2}.$$

Show that *I*¹ diverges.]

54. Show that the Laplace transform $\mathscr{L} \{2te^{t^2} \cos e^{t^2}\}$ exists. [*Hint*: Start with integration by parts.]

55. If $\mathscr{L}{f(t)} = F(s)$ and a = 0 is a constant, show that

$$\mathscr{L}\left\{f(at)\right\} = \frac{1}{a}F\left(\frac{s}{a}\right)$$

This result is known as the change of scale theor em.

56. Use the given Laplace transform and the result in Problem 55 to find the indicated Laplace transform. Assume that a and k are positive constants.

(a)
$$\mathscr{L} \{e^t\} = \frac{1}{s-1}$$
; $\mathscr{L} \{e^{at}\}$
(b) $\mathscr{L} \{\sin t\} = \frac{1}{s^2+1}$; $\mathscr{L} \{\sin kt\}$

(c)
$$\mathscr{L}\{1 - \cos t\} = \frac{1}{s(s^2 + 1)}; \quad \mathscr{L}\{1 - \cos kt\}$$

(d) $\mathscr{L} \{ \sin t \sinh t \} = \frac{2s}{s^4 + 4}; \quad \mathscr{L} \{ \sin kt \sinh kt \}$

7.2 INVERSE TRANSFORMS AND TRACEORMS OF DERIVATIVES
EVIEW MATERIAL
Partial fraction decomposition
Sende staticative Manual
INTRODUCTION of this section we take a few small steps into an investigation of how the Laplace transform can be used to solve certain types of equations for an unknown function. We big in the discussion with the concept of the inverse Laplace transform or, more precisely, the inverse of a Laplace transform *F(s)*. After some important preliminary background material on the Laplace transform of derivatives *f'(t)*, *f''(t)*, ..., we then illustrate how both the Laplace transform and the in-

7.2.1 INVERSE TRANSFORMS

verse Laplace transform come into play in solving some simple ordinary differential equations.

The Inverse Problem If F(s) represents the Laplace transform of a function f(t), that is, $\mathscr{L}{f(t)} = F(s)$, we then say f(t) is the **inverse Laplace transform** of F(s) and write $f(t) = \mathscr{L}^{-1}{F(s)}$. For example, from Examples 1, 2, and 3 of Section 7.1 we have, respectively,

Transform	Inverse Transform				
$\mathcal{L}\{1\} = \frac{1}{s}$	$1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$				
$\mathscr{L}{t} = \frac{1}{s^2}$	$t = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}$				
$\mathscr{L}\{e^{-3t}\} = \frac{1}{s+3}$	$e^{-3t} = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$				

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and thus, from the linearity of \mathcal{L}^{-1} and part (c) of Theorem 7.2.1,

$$\mathscr{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = -\frac{16}{5}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\}$$
$$= -\frac{16}{5}e^t + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}.$$
(5)

7.2.2 TRANSFORMS OF DERIVATIVES

Transform a Derivative As was pointed out in the introduction to this chapter, our immediate goal is to use the Laplace transform to solve differential equations. To that end we need to evaluate quantities such as $\mathcal{L}\{dy/dt\}$ and $\mathcal{L}\{d^2y/dt^2\}$. For example, if f' is continuous for $t \ge 0$, then integration by parts gives

$$\mathcal{L}\lbrace f'(t)\rbrace = \int_0^\infty e^{-st} f'(t) \, dt = e^{-st} f(t) \Big|_0^\infty + s \int_0^\infty e^{-st} f(t) \, dt$$
$$= -f(0) + s \mathcal{L}\lbrace f(t)\rbrace$$
$$\mathcal{L}\lbrace f'(t)\rbrace = sF(s) - f(0). \tag{6}$$

Here we have assumed that $e^{-st} f(t) \rightarrow 0$ as $t \rightarrow \infty$. Similarly, with the aid of (6),

$$\mathscr{L}\{f''(t)\} = \int_{0}^{\infty} e^{-st} f''(t) dt = e^{-st} f'(t) \Big|_{0}^{\infty} + s \int_{0}^{\infty} e^{-st} f'(t) dt$$

$$= -f'(0) + s \mathscr{L}\{f'(t)\}$$

$$= s[sF(s) - j(0)] - j'(f) \leftarrow \text{from (6)}$$
or
$$\mathscr{L}\{f''(t)\} \equiv 2^{s} (-s) - j'(0) - f'(0). \qquad (7)$$
In like many set we be shown that
$$\mathscr{L}\{e^{st}(\tau) = -s^{s} f(s) - s^{2} f(0) - s f'(0) - f''(0). \qquad (8)$$

 $\mathscr{L}\{ \bigcup_{i=1}^{m} f_{i} = s_{i} f_{i}(s) - s_{i}^{2} f(0) - s_{i} f'(0) - f''(0).$ (8) The recursive nature of the Laplace transform of the derivatives of a function fof the deb apparent from the results in (6), (7), and (8). The next theorem gives the Daplace transform of the *n*th derivative of *f*. The proof is omitted.

THEOREM 7.2.2 Transform of a Derivative

If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}{f^{(n)}(t)} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathcal{L}{f(t)}$.

or

Solving Linear ODEs It is apparent from the general result given in Theorem 7.2.2 that $\mathcal{L}\{d^ny/dt^n\}$ depends on $Y(s) = \mathcal{L}\{y(t)\}$ and the n - 1 derivatives of y(t) evaluated at t = 0. This property makes the Laplace transform ideally suited for solving linear initial-value problems in which the differential equation has *constant coefficients* Such a differential equation is simply a linear combination of terms $y, y', y'', \dots, y^{(n)}$:

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_0 y = g(t),$$

$$y(0) = y_0, y'(0) = y_1, \dots, y^{(n-1)}(0) = y_{n-1},$$

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$$y(t) = \frac{1}{6}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2}\right\} - \frac{2}{3\sqrt{2}}\mathscr{L}^{-1}\left\{\frac{\sqrt{2}}{(s+2)^2+2}\right\}$$
$$= \frac{1}{6} + \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t}\cos\sqrt{2}t - \frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t.$$

7.3.2 TRANSLATION ON THE *t*-AXIS

Unit Step Function In engineering, one frequently encounters functions that are either "off" or "on." For example, an external force acting on a mechanical system or a voltage impressed on a circuit can be turned off after a period of time. It is convenient, then, to define a special function that is the number 0 (off) up to a certain time t = a and then the number 1 (on) after that time. This function is called the **unit step function** or the **Heaviside function**, named after the English polymath **Oliver Heaviside** (1850–1925).

DEFINITION 7.3.1 Unit Step Function

The unit step function $\mathcal{U}(t - a)$ is defined to b

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a. \end{cases}$$

Notice that we define $\mathcal{U}(t-a)$ only on the conceptive *t*-axis, since this is all that we are concerned with in the and (a) the Caplace transform. In a broader sense $\mathcal{U}(t-a) = 0$ for t < c. The captor $\mathcal{U}(t-a)$ is given in Figure 7.3.2. In the case when a = 0 we can $\mathcal{U}(t) = 1$ for $t \ge 0$.

the function f defined for $t \ge 0$ is multiplied by $\mathcal{U}(t - a)$, the unit step function "turns off" a domain of the graph of that function. For example, consider the function f(x) + 2t - 3. To "turn off" the portion of the graph of f for $0 \le t < 1$, we almost found the product $(2t - 3)\mathcal{U}(t - 1)$. See Figure 7.3.3. In general, the graph of $f(t) \mathcal{U}(t - a)$ is 0 (off) for $0 \le t < a$ and is the portion of the graph of f (on) for $t \ge a$.

The unit step function can also be used to write piecewise-defined functions in a compact form. For example, if we consider $0 \le t < 2$, $2 \le t < 3$, and $t \ge 3$ and the corresponding values of $\mathcal{U}(t-2)$ and $\mathcal{U}(t-3)$, it should be apparent that the piecewise-defined function shown in Figure 7.3.4 is the same as $f(t) = 2 - 3\mathcal{U}(t-2) + \mathcal{U}(t-3)$. Also, a general piecewise-defined function of the type

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases}$$
(9)

is the same as

$$f(t) = g(t) - g(t) \mathcal{U}(t-a) + h(t) \mathcal{U}(t-a).$$
(10)

Similarly, a function of the type

$$f(t) = \begin{cases} 0, & 0 \le t < a \\ g(t), & a \le t < b \\ 0, & t \ge b \end{cases}$$
(11)

FIGURE 7.3.4 Function is $f(t) = 2 - 3\mathcal{U}(t-2) + \mathcal{U}(t-3)$

can be written

$$f(t) = g(t)[\mathcal{U}(t-a) - \mathcal{U}(t-b)].$$
(12)

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FIGURE 7.3.3 Function is $f(t) = (2t - 3) \mathcal{U}(t - 1)$

f(t)



SOLUTION Recall that because the beam is embedded at both ends, the boundary conditions are y(0) = 0, y'(0) = 0, y(L) = 0, y'(L) = 0. Now by (10) we can express w(x) in terms of the unit step function:

$$w(x) = w_0 \left(1 - \frac{2}{L}x\right) - w_0 \left(1 - \frac{2}{L}x\right) \mathcal{U}\left(x - \frac{L}{2}\right)$$
$$= \frac{2w_0}{L} \left[\frac{L}{2} - x + \left(x - \frac{L}{2}\right) \mathcal{U}\left(x - \frac{L}{2}\right)\right].$$

Transforming (19) with respect to the variable x gives

$$EI(s^{4}Y(s) - s^{3}y(0) - s^{2}y'(0) - sy''(0) - y'''(0)) = \frac{2w_{0}}{L} \left[\frac{L/2}{s} - \frac{1}{s^{2}} + \frac{1}{s^{2}}e^{-Ls/2} \right]$$

or
$$s^{4}Y(s) - sy''(0) - y'''(0) = \frac{2w_{0}}{EIL} \left[\frac{L/2}{s} - \frac{1}{s^{2}} + \frac{1}{s^{2}}e^{-Ls/2} \right].$$

If we let $c_1 = y''(0)$ and $c_2 = y'''(0)$, then

$$Y(s) = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{2w_0}{EIL} \left[\frac{L/2}{s^5} - \frac{1}{s^6} + \frac{1}{s^6} e^{-Ls/2} \right],$$

and consequently

$$y(x) = \frac{c_1}{2!} \mathscr{L}^{-1} \left\{ \frac{2!}{s^3} \right\} + \frac{c_2}{3!} \mathscr{L}^{-1} \left\{ \frac{3!}{s^4} \right\} + \frac{2w_0}{EIL} \left[\frac{L/2}{4!} \mathscr{L}^{-1} \left\{ \frac{4!}{s^5} \right\} - \frac{1}{5!} \mathscr{L}^{-1} \left\{ \frac{5!}{s^6} \right\} + \frac{1}{5!} \mathscr{L}^{-1} \left\{ \frac{5!}{s^6} e^{-Ls/2} \right\}$$
$$= \frac{c_1}{2} x^2 + \frac{c_2}{6} x^3 + \frac{w_0}{60 EIL} \left[\frac{5L}{2} x^4 - x^5 + \left(x - \frac{L}{5} \right)^5 \psi(x) \frac{L}{2} \right].$$

Applying the conditions y(L) = 0 to the last result yields a system of equations for L and 2.

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$$c_2 \frac{L^3}{6} + \frac{49w_0L^4}{1920EI} = 0$$

 $c_1 L + c_2 \frac{L^2}{2} + \frac{85w_0L^3}{960EI} = 0.$

Solving, we find $c_1 = 23w_0L^2/(960EI)$ and $c_2 = -9w_0L/(40EI)$. Thus the deflection is given by

$$y(x) = \frac{23w_0L^2}{1920EI}x^2 - \frac{3w_0L}{80EI}x^3 + \frac{w_0}{60EIL}\left[\frac{5L}{2}x^4 - x^5 + \left(x - \frac{L}{2}\right)^5\mathcal{U}\left(x - \frac{L}{2}\right)\right] .$$

EXERCISES 7.3

Answers to selected odd-numbered problems begin on page ANS-12.

7.3.1 TRANSLATION ON THE S-AXIS

In Problems 1–20 find either F(s) or f(t), as indicated.

- **1.** $\mathscr{L}\{te^{10t}\}$ **2.** $\mathscr{L}\{te^{-6t}\}$
- **3.** $\mathscr{L}\{t^3 e^{-2t}\}$ **4.** $\mathscr{L}\{t^{10} e^{-7t}\}$
- **5.** $\mathscr{L}{t(e^t + e^{2t})^2}$ **6.** $\mathscr{L}{e^{2t}(t-1)^2}$
- **7.** $\mathscr{L}\{e^t \sin 3t\}$ **8.** $\mathscr{L}\{e^{-2t} \cos 4t\}$
- **9.** $\mathscr{L}\{(1 e^t + 3e^{-4t})\cos 5t\}$
- $10. \ \mathscr{L}\left\{e^{3t}\left(9-4t+10\sin\frac{t}{2}\right)\right\}$

11. $\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^3}\right\}$	12. $\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^4}\right\}$
13. $\mathscr{L}^{-1}\left\{\frac{1}{s^2-6s+10}\right\}$	14. $\mathscr{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$
$15. \ \mathscr{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\}$	16. $\mathscr{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\}$
$17. \ \mathscr{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\}$	$18. \ \mathscr{L}^{-1}\left\{\frac{5s}{(s-2)^2}\right\}$
19. $\mathscr{L}^{-1}\left\{\frac{2s-1}{s^2(s+1)^3}\right\}$	20. $\mathscr{L}^{-1}\left\{\frac{(s+1)^2}{(s+2)^4}\right\}$



70. $y'' + 4y' + 3y = 1 - \mathcal{U}(t-2) - \mathcal{U}(t-4) + \mathcal{U}(t-6),$ y(0) = 0, y'(0) = 0

- 71. Suppose a 32-pound weight stretches a spring 2 feet. If the weight is released from rest at the equilibrium position, find the equation of motion x(t) if an impressed force f(t) = 20t acts on the system for $0 \le t < 5$ and is then removed (see Example 5). Ignore any damping forces. Use a graphing utility to graph x(t) on the interval [0, 10].
- **72.** Solve Problem 71 if the impressed force $f(t) = \sin t$ acts on the system for $0 \le t < 2\pi$ and is then removed.

In Problems 73 and 74 use the Laplace transform to find the charge q(t) on the capacitor in an *RC*-series circuit subject to the given conditions.

73. q(0) = 0, $R = 2.5 \Omega$, C = 0.08 f, E(t) given in Figure 7.3.19



FIGURE 7.3.19 *E*(*t*) in Problem 73

74. $q(0) = q_0$, $R = 10 \Omega$, C = 0.1 f, E(t) given in Figure 7.3.20



FIGURE 7.3.20 *E*(*t*) in Problem 74

- 75. (a) Use the Laplace transform to find the current i(t) in a single-loop LR-series circuit when i(0) = 0, L = 1 h, $R = 10 \Omega$, and E(t) is as given in Figure 7.3.21.
 - (b) Use a computer graphing program to graph i(t) for $0 \le t \le 6$. Use the graph to estimate i_{max} and i_{\min} , the maximum and minimum values of the current.





- 76. (a) Use the Laplace transform to find the charge q(t)on the capacitor in an RC-series circuit when $q(0) = 0, R = 50 \Omega, C = 0.01$ f, and E(t) is as given in Figure 7.3.22.
 - (b) Assume that $E_0 = 100$ V. Use a computer graphing program to graph q(t) for $0 \le t \le 6$. Use the graph to estimate q_{max} , the maximum value of the charge.



FIGURE 7.3.22 *E*(*t*) in Problem 76

77. A cantilever beam is embedded at its left end and free at its right end. Use the Laplace transform to find the deflection y(x) when the load is given by

$$w(x) = \begin{cases} w_0, & 0 < x < L/2 \\ 0, & L/2 \le x < L. \end{cases}$$

78. Solve Problem 77 ymn the oud is given by

$$x) = \begin{cases} 0, & 0 < x < L/2 \\ w_0, & L/3 < x < 2L/2 \\ 0, & 2L/3 < x < L \end{cases}$$

1 f, E(t) given in NOTES $W(x) = \begin{cases} 0, & 0 < x < L/3 \\ w_0, & L/3 < x < 2L/3 \\ 0, & 2L/3 < x < L. \end{cases}$ Find the deflection y(x) of a cantilever beam embedded at its left end and free at its right end when the load is consistent. given in Example 10.

> 80. A beam is embedded at its left end and simply supported at its right end. Find the deflection y(x) when the load is as given in Problem 77.

Mathematical Model

- 81. Cake Inside an Oven Reread Example 4 in Section 3.1 on the cooling of a cake that is taken out of an oven.
 - (a) Devise a mathematical model for the temperature of a cake while it is *inside* the oven based on the following assumptions: At t = 0 the cake mixture is at the room temperature of 70°; the oven is not preheated, so at t = 0, when the cake mixture is placed into the oven, the temperature inside the oven is also 70°; the temperature of the oven increases linearly until t = 4 minutes, when the desired temperature of 300° is attained; the oven temperature is a constant 300° for $t \ge 4$.
 - (b) Use the Laplace transform to solve the initial-value problem in part (a).

Inverse Form of Theorem 7.4.2 The convolution theorem is sometimes useful in finding the inverse Laplace transform of the product of two Laplace transforms. From Theorem 7.4.2 we have

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f * g.$$
(4)

Many of the results in the table of Laplace transforms in Appendix III can be derived using (4). For example, in the next example we obtain entry 25 of the table:

$$\mathscr{L}\{\sin kt - kt \cos kt\} = \frac{2k^3}{(s^2 + k^2)^2}.$$
 (5)

EXAMPLE 4 Inverse Transform as a Convolution

Evaluate $\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}$ **SOLUTION** Let $F(s) = G(s) = \frac{1}{s^2 + k^2}$ so that $f(t) = g(t) = \frac{1}{k} \mathcal{L}^{-1} \left\{ \frac{k}{s^2 + k^2} \right\} = \frac{1}{k} \sin kt.$ In this case (4) gives $\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{k^2} \int_0^t \sin k\tau \sin k(t-\tau) \, d\tau.$ (6)With the aid of the product-to-sum trigonometric identity $\sin A \sin B = \frac{1}{2} [\cos(A - B - \cos(A + B))]$ and the substitutions $A = k \tan(A - \tau)$ we can carry out the integration in (6): Preview from Noti Preview from 354 0 Page 354 0 $= \frac{1}{2k^2} \left[\frac{1}{2k} \sin k(2\tau - t) - \cos kt \right] d\tau$ $= \frac{1}{2k^2} \left[\frac{1}{2k} \sin k(2\tau - t) - \tau \cos kt \right]_{0}^{t}$ $=\frac{\sin kt - kt\cos kt}{2k^3}.$ _

Multiplying both sides by $2k^3$ gives the inverse form of (5).

Transform of an Integral When g(t) = 1 and $\mathscr{L}{g(t)} = G(s) = 1/s$, the convolution theorem implies that the Laplace transform of the integral of f is

$$\mathscr{L}\left\{\int_{0}^{t} f(\tau) d\tau\right\} = \frac{F(s)}{s}.$$
(7)

The inverse form of (7),

$$\int_{0}^{t} f(\tau) d\tau = \mathcal{L}^{-1} \left\{ \frac{F(s)}{s} \right\},$$
(8)

can be used in lieu of partial fractions when s^n is a factor of the denominator and $f(t) = \mathcal{L}^{-1}{F(s)}$ is easy to integrate. For example, we know for $f(t) = \sin t$ that $F(s) = 1/(s^2 + 1)$, and so by (8)

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^{2}+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/(s^{2}+1)}{s}\right\} = \int_{0}^{t} \sin\tau d\tau = 1 - \cos t$$
$$\mathcal{L}^{-1}\left\{\frac{1}{s^{2}(s^{2}+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/s(s^{2}+1)}{s}\right\} = \int_{0}^{t} (1 - \cos\tau) d\tau = t - \sin t$$
$$\mathcal{L}^{-1}\left\{\frac{1}{s^{3}(s^{2}+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/s^{2}(s^{2}+1)}{s}\right\} = \int_{0}^{t} (\tau - \sin\tau) d\tau = \frac{1}{2}t^{2} - 1 + \cos\tau$$

and so on.

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Volterra Integral Equation The convolution theorem and the result in (7) are useful in solving other types of equations in which an unknown function appears under an integral sign. In the next example we solve a Volterra integral equation for f(t),

$$f(t) = g(t) + \int_0^t f(\tau) h(t - \tau) d\tau.$$
 (9)

The functions g(t) and h(t) are known. Notice that the integral in (9) has the convolution form (2) with the symbol h playing the part of g.

EXAMPLE 5 An Integral Equation
Solve
$$f(t) = 3t^2 - e^{-t} - \int_0^t f(\tau) e^{t-\tau} d\tau$$
 for $f(t)$.

SOLUTION In the integral we identify $h(t - \tau) = e^{t-\tau}$ so that $h(t) = e^t$. We take the Laplace transform of each term; in particular, by Theorem 7.4.2 the transform of the integral is the product of $\mathscr{L}{f(t)} = F(s)$ and $\mathscr{L}{e^t} = 1/(s-1)$:

$$F(s) = 3 \cdot \frac{2}{s^3} - \frac{1}{s+1} - F(s) \cdot \frac{1}{s-1}$$

After solving the last equation for F(s) and carrying out the partial fraction decomposition, we find



Series Circuits In a single-loop or series circuit, Kirchhoff's second law states that the sum of the voltage drops across an inductor, resistor, and capacitor is equal to the impressed voltage E(t). Now it is known that the voltage drops across an inductor, resistor, and capacitor are, respectively,

$$L \frac{di}{dt}$$
, $Ri(t)$, and $\frac{1}{C} \int_0^t i(\tau) d\tau$,

where i(t) is the current and L, R, and C are constants. It follows that the current in a circuit, such as that shown in Figure 7.4.2, is governed by the integrodifferential equation

$$L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau) d\tau = E(t).$$
(10)

EXAMPLE 6 **An Integrodifferential Equation**

Determine the current i(t) in a single-loop LRC-series circuit when L = 0.1 h, $R = 2 \Omega$, C = 0.1 f, i(0) = 0, and the impressed voltage is

$$E(t) = 120t - 120t \mathcal{U}(t-1).$$

L

FIGURE 7.4.2 LRC-series circuit

R

E





(*i*)
$$\delta(t - t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases}$$
 and (*ii*) $\int_0^\infty \delta(t - t_0) dt = 1.$

The unit impulse $\delta(t - t_0)$ is called the **Dirac delta function.**

It is possible to obtain the Laplace transform of the Dirac delta function by the formal assumption that $\mathscr{L}{\delta(t-t_0)} = \lim_{a\to 0} \mathscr{L}{\delta_a(t-t_0)}.$



(b) behavior of δ_a as $a \to 0$



THEOREM 7.5.1 Transform of the Dirac Delta Function For t_0 $\mathscr{L}\{\delta(t-t_0)\}=e^{-st_0}.$ 0. (3)

PROOF To begin, we can write $\delta_a(t - t_0)$ in terms of the unit step function by virtue of (11) and (12) of Section 7.3:

$$\delta_a(t-t_0) = \frac{1}{2a} [\mathcal{U}(t-(t_0-a)) - \mathcal{U}(t-(t_0+a))].$$

By linearity and (14) of Section 7.3 the Laplace transform of this last expression is

$$\mathscr{L}\{\delta_a(t-t_0)\} = \frac{1}{2a} \left[\frac{e^{-s(t_0-a)}}{s} - \frac{e^{-s(t_0+a)}}{s} \right] = e^{-st_0} \left(\frac{e^{sa} - e^{-sa}}{2sa} \right).$$
(4)

Since (4) has the indeterminate form 0/0 as $q \rightarrow 1$ we apply L'Hôpital's Rule:

$$\mathscr{L}\{\delta(t-t_0)\} = \lim_{a \to 0} \mathscr{L}\{\delta(t-t_0)\} = e^{-st_0} \lim_{a \to 0} \left(\frac{e^{sa} - e^{-sa}}{2sa}\right) = e^{-st_0}.$$
Now that $t_0 = 0$, it seens blockible to conclude from (3) that
$$\mathscr{L}\{\delta(t)\} = 1.$$
Solution that we have been considering, since we expect from Theorem 7.1.3 that $\mathscr{L}\{f(t)\} \to 0$ as $s \to \infty$.
EXAMPLE 1 Two Initial-Value Problems

2 β β second sufficient that $\delta(t)$ is not the usual type of function that we have seen considering, since we expect from Theorem 7.1.3 that $\mathscr{L}{f(t)} \to 0$ as $s \to \infty$.

EXAMPLE 1 **Two Initial-Value Problems**

Solve $y'' + y = 4 \delta(t - 2\pi)$ subject to

(a)
$$y(0) = 1$$
, $y'(0) = 0$ (b) $y(0) = 0$, $y'(0) = 0$.

The two initial-value problems could serve as models for describing the motion of a mass on a spring moving in a medium in which damping is negligible. At $t = 2\pi$ the mass is given a sharp blow. In (a) the mass is released from rest 1 unit below the equilibrium position. In (b) the mass is at rest in the equilibrium position.

SOLUTION (a) From (3) the Laplace transform of the differential equation is

$$s^{2}Y(s) - s + Y(s) = 4e^{-2s}$$
 or $Y(s) = \frac{s}{s^{2} + 1} + \frac{4e^{-2s}}{s^{2} + 1}$.

Using the inverse form of the second translation theorem, we fin

$$y(t) = \cos t + 4\sin(t-2) \mathcal{U}(t-2)$$

Since $sin(t - 2\pi) = sin t$, the foregoing solution can be written as

$$y(t) = \begin{cases} \cos t, & 0 \le t < 2\\ \cos t + 4 \sin t, & t \ge 2 \end{cases}.$$
(5)

Answers to selected odd-numbered problems begin on page ANS-13.

EXERCISES 7.6

In Problems 1-12 use the Laplace transform to solve the given system of differential equations.
1.
$$\frac{dx}{dt} = -x + y$$
 2. $\frac{dx}{dt} = 2y + e^i$
 $\frac{dy}{dt} = 2x$ $\frac{dy}{dt} = 2y + e^i$
 $\frac{dy}{dt} = 2x$ $\frac{dy}{dt} = 8x - i$
 $x(0) = 0, y(0) = 1$ $x(0) = 1, y(0) = 1$
3. $\frac{dx}{dt} = x - 2y$ 4. $\frac{dx}{dt} + 3x + \frac{dy}{dt} = 1$
 $\frac{dy}{dt} = 5x - y$ $\frac{dx}{dt} + 3x + \frac{dy}{dt} = 1$
 $\frac{dx}{dt} - x + \frac{dy}{dt} - y = e^i$
 $x(0) = 0, y(0) = 1$ $x(0) = 0, y(0) = 0$
5. $2\frac{dx}{dt} + \frac{dy}{dt} - 2x = 1$
 $\frac{dx}{dt} + \frac{dy}{dt} - 2x = 1$
 $\frac{dx}{dt} + \frac{dy}{dt} - 2x = 1$
 $\frac{dx}{dt} + \frac{dy}{dt} - 3x - 3y = 2$
 $x(0) = 0, y(0) = 0$
6. $\frac{dx}{dt} + \frac{dy}{dt} - 2x = 1$
 $\frac{dx}{dt} + \frac{dy}{dt} + 2y = 0$
 $x(0) = 0, y(0) = 1$
7. $\frac{d^2x}{dt^2} + x - y = 0$
 $y(0) = 0, y(0) = 1$
7. $\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} = i$
 $y(0) = 0, y(0) = 1$
7. $\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} = i$
 $y(0) = 0, y(0) = 1$
 $\frac{dx}{dt} + \frac{dy}{dt^2} = i^2$
 $y(0) = 0, y(0) = 0$, $y(0) = -1, x(0) = 0, y(0) = 0$
 $y(0) = 0, y(0) = 1$
 $\frac{dx}{dt} + \frac{dy}{dt^2} = i^2$
 $y(0) = 0, y(0) = 0$
 $\frac{dx}{dt} + \frac{dy}{dt^2} = i^2$
 $x(0) = 0, y(0) = 0$, $y(0) = 0$
 $\frac{dx}{dt} + \frac{dy}{dt^2} = i^2$
 $x(0) = 0, y(0) = 0, y(0) = 0$
 $\frac{dx}{dt} + \frac{dy}{dt^2} = i^2$
 $x(0) = 0, y(0) = 0, y(0) = 0$
 $\frac{dx}{dt} + \frac{dy}{dt^2} = i^2$
 $x(0) = 0, y(0) = 0, y(0) = 0$
 $\frac{dx}{dt} + \frac{dy}{dt^2} = i^2$
 $\frac{dx}{dt} + 2x - 2\frac{d^2y}{dt^2} = 0$
 $x(0) = 0, y(0) = 0, y(0) = 0, y(0) = 0$
 $\frac{dx}{dt^2} + \frac{dy}{dt^2} + 3y = 1e^{-1}$
 $x(0) = 0, y(0) = 0, y(0) = 0$
 $\frac{dx}{dt} + \frac{dy}{dt} + 3y = 1e^{-1}$
 $x(0) = 0, y(0) = 0, y(0) = 0$
 $\frac{dx}{dt} + \frac{dy}{dt} + 3y = 1e^{-1}$
 $x(0) = 0, y(0) = 0, y(0) = 0$
 $\frac{dx}{dt} + \frac{dy}{dt} + \frac{dy}{dt} + \frac{dx}{dt} + 2x - 2\frac{d^2y}{dt^2} = 0$
 $x(0) = 0, y(0) = 0, x(0) = 0, y(0) = 0$
 $\frac{dx}{dt} + \frac{dy}{dt} + \frac{dx}{dt} + 2x - 2\frac{dy}{dt^2} = 0$
 $x(0) = 0, y(0) = 0, y(0) = 0$
 $\frac{dx}{dt} + \frac{dy}{dt} + \frac{dx}{$



FIGURE 7.R.1 Graph for Problems 25–28











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32.

8.1 PRELIMINARY THEORY—LINEAR SYSTEMS

REVIEW MATERIAL

• Matrix notation and properties are used extensively throughout this chapter. It is imperative that you review either Appendix II or a linear algebra text if you unfamiliar with these concepts.

INTRODUCTION Recall that in Section 4.9 we illustrated how to solve systems of n linear differential equations in n unknowns of the form

$$P_{11}(D)x_{1} + P_{12}(D)x_{2} + \dots + P_{1n}(D)x_{n} = b_{1}(t)$$

$$P_{21}(D)x_{1} + P_{22}(D)x_{2} + \dots + P_{2n}(D)x_{n} = b_{2}(t)$$

$$\vdots$$

$$P_{n1}(D)x_{1} + P_{n2}(D)x_{2} + \dots + P_{nn}(D)x_{n} = b_{n}(t),$$
(1)

where the P_{ij} were polynomials of various degrees in the differential operator D. In this chapter we confine our study to systems of first-order DEs that are special cases of systems that have the normal form

$$\frac{dx_1}{dt} = g_1(t, x_1, x_2, \dots, x_n)$$

$$\frac{dx_2}{dt} = g_2(t, x_1, x_2, \dots, x_n)$$

$$\vdots$$

$$\frac{dx_n}{dt} = g_n(t, x_1, x_2, \dots, x_n)$$

$$\frac{dx_n}{dt} = g_n(t, x_1, x_2, \dots, x_n)$$
(2)
A system such as (2) of *n* first-output turns is called a **first-orde** system.
$$Q = \text{Linear. Systems} \quad \text{(Linear for the functions } g_1, g_2, \dots, g_n \text{ in } (2) \text{ is linear in the degeneen variables } x_1, x_2, \dots, x_n, \text{ we get the normal form of a first-orde system of nnear equations:}$$

$$\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n + f_1(t)$$

$$\frac{ax_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n + f_1(t)$$

$$\frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n + f_2(t)$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n + f_n(t).$$
(3)

We refer to a system of the form given in (3) simply as a **linear system.** We assume that the coefficients a_{ij} as well as the functions f_i are continuous on a common interval *I*. When $f_i(t) = 0$, i = 1, 2, ..., n, the linear system (3) is said to be **homogeneous;** otherwise, it is **nonhomogeneous.**

Matrix Form of a Linear System If \mathbf{X} , $\mathbf{A}(t)$, and $\mathbf{F}(t)$ denote the respective matrices

$$\mathbf{X} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \qquad \mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \qquad \mathbf{F}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix},$$

EXAMPLE 1

Distinct Eigenvalues

Solve

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = 2x + y.$$
(4)

SOLUTION We first find the eigenvalues and eigenvectors of the matrix of coefficients

From the characteristic equation

$$\det(\mathbf{A} - \mathbf{I}) = \begin{vmatrix} 2 & - & 3 \\ 2 & 1 & - \end{vmatrix} = \begin{vmatrix} 2 & - & 3 \end{vmatrix} = \begin{vmatrix} 2 & - & 3 \end{vmatrix} = -4 = (-4)(-4) = 0$$

we see that the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 4$. Now for $\lambda_1 = -1$, (3) is equivalent to

$$3k_1 + 3k_2 = 0$$

$$2k_1 + 2k_2 = 0.$$

Thus $k_1 = -k_2$. When $k_2 = -1$, the related eigenvector is 6 5 4 3 2 For $_{2} = 4$ we have 1 -12 -3 -21 3 (a) graph of $x = e^{-t} + 3e^{4t}$ pm

(b) graph of
$$y = -e^{-t} + 2e^{4}$$

1 2 3

3 $^{-2}$ -1



⁽c) trajectory define by $x = e^{-t} + 3e^{4t}, y = -e^{-t} + 2e^{4t}$ in the phase plane

FIGURE 8.2.1 A solution from (5) yields three different curves in three different planes

natrix of coefficients A is a 2×2 matrix and since we have found two linly independent solutions of (4),

$$\mathbf{X}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$
 and $\mathbf{X}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{4t}$,

we conclude that the general solution of the system is

$$\mathbf{X} = c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 = c_1 \binom{1}{-1} e^{-t} + c_2 \binom{3}{2} e^{4t}.$$
 (5)

the corresponding eigenvector is

Phase Portrait You should keep firmly in mind that writing a solution of a system of linear first-order differential equations in terms of matrices is simply an alternative to the method that we employed in Section 4.9, that is, listing the individual functions and the relationship between the constants. If we add the vectors on the right-hand side of (5) and then equate the entries with the corresponding entries in the vector on the left-hand side, we obtain the more familiar statement

$$x = c_1 e^{-t} + 3c_2 e^{4t}, \qquad y = -c_1 e^{-t} + 2c_2 e^{4t}$$

As was pointed out in Section 8.1, we can interpret these equations as parametric equations of curves in the xy-plane or phase plane. Each curve, corresponding to specific choices for c_1 and c_2 , is called a **trajectory**. For the choice of constants $c_1 = c_2 = 1$ in the solution (5) we see in Figure 8.2.1 the graph of x(t) in the *tx*-plane, the graph of y(t) in the *ty*-plane, and the trajectory consisting of the points

Computer Lab Assignments

In Problems 15 and 16 use a CAS or linear algebra software as an aid in finding the general solution of the given system.

15.
$$\mathbf{X}' = \begin{pmatrix} 0.9 & 2.1 & 3.2 \\ 0.7 & 6.5 & 4.2 \\ 1.1 & 1.7 & 3.4 \end{pmatrix} \mathbf{X}$$

16. $\mathbf{X}' = \begin{pmatrix} 1 & 0 & 2 & -1.8 & 0 \\ 0 & 5.1 & 0 & -1 & 3 \\ 1 & 2 & -3 & 0 & 0 \\ 0 & 1 & -3.1 & 4 & 0 \\ -2.8 & 0 & 0 & 1.5 & 1 \end{pmatrix} \mathbf{X}$

- 17. (a) Use computer software to obtain the phase portrait of the system in Problem 5. If possible, include arrowheads as in Figure 8.2.2. Also include four half-lines in your phase portrait.
 - (b) Obtain the Cartesian equations of each of the four half-lines in part (a).
 - (c) Draw the eigenvectors on your phase portrait of the system.
- **18.** Find phase portraits for the systems in Problems 2 and 4. For each system find any half-line trajectories and include these lines in your phase portrait.

REPEATED EIGENVALUES 8.2.2

In Problems 19–28 find the grant a system.

19
$$\frac{d}{dt} = 3x - y$$

$$\frac{dy}{dt} = 9x - 3y$$
21.
$$\mathbf{X}' = \begin{pmatrix} -1 & 3 \\ -3 & 5 \end{pmatrix} \mathbf{X}$$
22.
$$\mathbf{X}' = \begin{pmatrix} 12 & -9 \\ 4 & 0 \end{pmatrix} \mathbf{X}$$
23.
$$\frac{dx}{dt} = 3x - y - z$$

$$\frac{dy}{dt} = x + y - z$$

$$\frac{dy}{dt} = x - y + z$$

$$\frac{dz}{dt} = x - y + z$$

$$\frac{dz}{dt} = 4x + 2y + 3z$$

25.
$$\mathbf{X}' = \begin{pmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix} \mathbf{X}$$
 26. $\mathbf{X}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{X}$

27. $\mathbf{X}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix} \mathbf{X}$ **28.** $\mathbf{X}' = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix} \mathbf{X}$

In Problems 29 and 30 solve the given initial-value problem.

29.
$$\mathbf{X}' = \begin{pmatrix} 2 & 4 \\ -1 & 6 \end{pmatrix} \mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} -1 \\ 6 \end{pmatrix}$$

30. $\mathbf{X}' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$

31. Show that the 5×5 matrix

	2	1	0	0	0	
	0	2	0	0	0	
A =	0	0	2	0	0	
	0	0	0	2	1	
	0/	0	0	0	2/	

has an eigenvalue λ_1 of multiplicity 5. Show that three linearly independent eigenvectors corresponding to λ_1 can be found.

Computer Lab Assignments

32. Find phase portraits for the systems in Problems 20 and 21. For each system find any half-line trajectories and include these lines in your phase portrait.

estimate the seneral solution of the given system.

33.
$$\frac{dx}{dt} = 6x - y$$

 $\frac{dy}{dt} = 5x + 2y$
34. $\frac{dx}{dt} = x + y$
 $\frac{dy}{dt} = x + y$
 $\frac{dy}{dt} = -2x - y$
35. $\frac{dx}{dt} = 5x + y$
 $\frac{dy}{dt} = -2x + 3y$
36. $\frac{dx}{dt} = 4x + 5y$
 $\frac{dy}{dt} = -2x + 6y$
37. $\mathbf{X}' = \begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix} \mathbf{X}$
38. $\mathbf{X}' = \begin{pmatrix} 1 & -8 \\ 1 & -3 \end{pmatrix} \mathbf{X}$
39. $\frac{dx}{dt} = z$
 $\frac{dy}{dt} = -z$
 $\frac{dy}{dt} = -z$
 $\frac{dy}{dt} = -z$
 $\frac{dz}{dt} = y$
40. $\frac{dx}{dt} = 2x + y + 2z$
 $\frac{dy}{dt} = 3x + 6z$
 $\frac{dz}{dt} = -4x - 3z$
41. $\mathbf{X}' = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \mathbf{X}$
42. $\mathbf{X}' = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 6 & 0 \\ -4 & 0 & 4 \end{pmatrix} \mathbf{X}$

 $\sqrt{-1}$

43.
$$\mathbf{X}' = \begin{pmatrix} 2 & 5 & 1 \\ -5 & -6 & 4 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{X}$$
 44. $\mathbf{X}' = \begin{pmatrix} 2 & 4 & 4 \\ -1 & -2 & 0 \\ -1 & 0 & -2 \end{pmatrix} \mathbf{X}$

In Problems 45 and 46 solve the given initial-value problem.

45.
$$\mathbf{X}' = \begin{pmatrix} 1 & -12 & -14 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{pmatrix} \mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$$

46. $\mathbf{X}' = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix} \mathbf{X}, \quad \mathbf{X}(0) = \begin{pmatrix} -2 \\ 8 \end{pmatrix}$

Computer Lab Assignments

- **47.** Find phase portraits for the systems in Problems 36, 37, and 38.
- 48. (a) Solve (2) of Section 7.6 using the first method outlined in the *Remarks* (page 345)—that is, express (2) of Section 7.6 as a first-order system of four linear equations. Use a CAS or linear algebra software as an aid in finding eigenvalues and eigenvectors of a 4 × 4 matrix. Then apply the initial conditions to your general solution to obtain (4) of Section 7.6.
 - (b) Solve (2) of Section 7.6 using the second method outlined in the *Remarks*—that is, express (2) of Section 7.6 as a second-order system of two linear equations. Assume solutions of the form $\mathbf{X} = \mathbf{V} \sin \alpha$

and $\mathbf{X} = \mathbf{V} \cos \omega t$. Find the eigenvalues and eigenvectors of a 2 × 2 matrix. As in part (a), obtain (4) of Section 7.6.

Discussion Problems

49. Solve each of the following linear systems.

(a)
$$\mathbf{X}' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{X}$$
 (b) $\mathbf{X}' = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{X}$

Find a phase portrait of each system. What is the geometric significance of the line y = -x in each portrait?

- **50.** Consider the 5×5 matrix given in Problem 31. Solve the system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ without the aid of matrix methods, but write the general solution using matrix notation. Use the general solution as a basis for a discussion of how the system can be solved using the matrix methods of this section. Carry out your ideas.
- **51.** Obtain a Cartesian equation of the curve define parametrically by the solution of the linear system in Example 6. Identify the curve passing through (2, -1) in Figure 8.2.5. [*Hint*: Compute x^2 , y^2 , and xy.]
- 52. Examine your phase portraits in Problem 47. Under what conditions will the parse portrait of a 2 × 2 homogeneous linear system with complex eigenvalues consister a family or closed curves? consist of a family of sprass. Under what conditions is the origin (0, 0) a repeller? An attractor?

- Section 4.4 (Undetermined Coefficients
- Section 4.6 (Variation of Parameters)

INTRODUCTION In Section 8.1 we saw that the general solution of a nonhomogeneous linear system $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$ on an interval *I* is $\mathbf{X} = \mathbf{X}_c + \mathbf{X}_p$, where $\mathbf{X}_c = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 + \cdots + c_n\mathbf{X}_n$ is the **complementary function** or general solution of the associated homogeneous linear system $\mathbf{X}' = \mathbf{A}\mathbf{X}$ and \mathbf{X}_p is any **particular solution** of the nonhomogeneous system. In Section 8.2 we saw how to obtain \mathbf{X}_c when the coefficient matrix \mathbf{A} was an $n \times n$ matrix of constants. In the present section we consider two methods for obtaining \mathbf{X}_p .

The methods of **undetermined coefficient** and **variation of parameters** used in Chapter 4 to find particular solutions of nonhomogeneous linear ODEs can both be adapted to the solution of nonhomogeneous linear systems $\mathbf{X}' = \mathbf{A}\mathbf{X} + \mathbf{F}(t)$. Of the two methods, variation of parameters is the more powerful technique. However, there are instances when the method of undetermined coefficients provides a quick means of finding a particular solutio

8.3.1 UNDETERMINED COEFFICIENTS

The Assumptions As in Section 4.4, the method of undetermined coefficient consists of making an educated guess about the form of a particular solution vector \mathbf{X}_{p} ; the guess is motivated by the types of functions that make up the entries of the

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- **21.** Suppose $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where **D** is defined as in (9). Use (3) to show that $e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}$.
- **22.** If **D** is defined as in (9), then fin $e^{\mathbf{D}t}$.

In Problems 23 and 24 use the results of Problems 19–22 to solve the given system.

23.
$$\mathbf{X}' = \begin{pmatrix} 2 & 1 \\ -3 & 6 \end{pmatrix} \mathbf{X}$$

24. $\mathbf{X}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{X}$

Discussion Problems

for k =

- 25. Reread the discussion leading to the result given in (7). Does the matrix sI - A always have an inverse? Discuss.
- 26. A matrix A is said to be nilpotent if there exists some positive integer *m* such that $A^m = 0$. Verify that

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 1\\ -1 & 0 & 1\\ -1 & 1 & 1 \end{pmatrix}$$

is nilpotent. Discuss why it is relatively easy to compute Use MATLABOR OS to find e^A CHAPTER 8 IN REDAM Ims 1 co 2 fil Cherblor 1 $e^{\mathbf{A}t}$ when **A** is nilpotent. Compute $e^{\mathbf{A}t}$ and then use (1) to solve the system $\mathbf{X}' = \mathbf{A}\mathbf{X}$.

Computer Lab Assignments

- 27. (a) Use (1) to find the general solution of $\mathbf{X}' = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} \mathbf{X}$. Use a CAS to find $e^{\mathbf{A}t}$. Then use the computer to find eigenvalues and eigenvectors of the coefficient matrix $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$ and form the general solution in the manner of Section 8.2. Finally, reconcile the two forms of the general solution of the system. (b) Use (1) to find the general solution of
 - $\mathbf{X}' = \begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{X}$. Use a CAS to find $e^{\mathbf{A}t}$. In the

case of complex output, utilize the software to do the simplification; for example, in Mathematica, if $\mathbf{m} = \mathbf{MatrixExp}[\mathbf{A} \ \mathbf{t}]$ has complex entries, then try the command Simplify[ComplexExpand[m]].

28. Use (1) to find the general solution o

$$\mathbf{X}' = \begin{pmatrix} -4 & 0 & 6 & 0\\ 0 & -5 & 0 & -4\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 2 \end{pmatrix} \mathbf{X}.$$

In Problems 1 and 2 first the blanks 1. The vector $\mathbf{X} = k \begin{pmatrix} 4 \\ 5 \end{pmatrix}$ is a solution of

the general solution of the system if it is known that

$$\lambda_1 = 1 + 2i$$
 is an eigenvalue and $\mathbf{K}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ is a corre-

sponding eigenvector?

In Problems 5-14 solve the given linear system.

5.
$$\frac{dx}{dt} = 2x + y$$

 $\frac{dy}{dt} = -x$
6. $\frac{dx}{dt} = -4x + 2y$
 $\frac{dy}{dt} = 2x - 4y$
7. $\mathbf{X}' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \mathbf{X}$
8. $\mathbf{X}' = \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix} \mathbf{X}$
9. $\mathbf{X}' = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 4 & 3 & 1 \end{pmatrix} \mathbf{X}$
10. $\mathbf{X}' = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix} \mathbf{X}$

2. The vector $\mathbf{X} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-9t} + c_2 \begin{pmatrix} 5 \\ 3 \end{pmatrix} e^{7t}$ is solution of the initial-value problem $\mathbf{X}' = \begin{pmatrix} 1 & 10 \\ 6 & -3 \end{pmatrix} \mathbf{X}, \mathbf{X}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ for $c_1 = _$ _____ and $c_2 = _$ _____

 $\mathbf{X}' = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} \mathbf{X} - \begin{pmatrix} 8 \\ 1 \end{pmatrix}$

3. Consider the linear system $\mathbf{X}' = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix} \mathbf{X}.$

Without attempting to solve the system, determine which one of the vectors

$$\mathbf{K}_1 = \begin{pmatrix} 0\\1\\1 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 1\\1\\-1 \end{pmatrix}, \quad \mathbf{K}_3 = \begin{pmatrix} 3\\1\\-1 \end{pmatrix}, \quad \mathbf{K}_4 = \begin{pmatrix} 6\\2\\-5 \end{pmatrix}$$

a calculator that uses base 10 arithmetic and carries four digits, so that $\frac{1}{3}$ is represented in the calculator as 0.3333 and $\frac{1}{9}$ is represented as 0.1111. If we use this calculator to compute $\left(x^2 - \frac{1}{9}\right) / \left(x - \frac{1}{3}\right)$ for x = 0.3334, we obtain

$$\frac{(0.3334)^2 - 0.1111}{0.3334 - 0.3333} = \frac{0.1112 - 0.1111}{0.3334 - 0.3333} = 1.$$

With the help of a little algebra, however, we see that

$$\frac{x^2 - \frac{1}{9}}{x - \frac{1}{3}} = \frac{\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right)}{x - \frac{1}{3}} = x + \frac{1}{3},$$

so when x = 0.3334, $\left(x^2 - \frac{1}{9}\right) / \left(x - \frac{1}{3}\right) \approx 0.3334 + 0.3333 = 0.6667$. This example shows that the effects of round-off error can be quite serious unless some care is taken. One way to reduce the effect of round-off error is to minimize the number of calculations. Another technique on a computer is to use double-precision arithmetic to check the results. In general, round-off error is unpredictable and difficult to analyze, and we will neglect it in the error analysis that follows. We will concentrate on investigating the error introduced by using a formula or algorithm to approximate the values of the solution.

Truncation Errors for Euler's Method In the sequence of values y_1, y_2 , y_3, \ldots generated from (1), usually the value of y_1 will not agree with the actual solution at x_1 —namely, $y(x_1)$ —because the algorithm gives only a straight-line approximation to the solution. See Figure 2.6.2. The error is called the local truncation

error, formula error, or discretization error. It occurs at each step; that is, if we assume that y_n is accurate, then y_{n+1} will contain local truncation error. To derive a formula for the local truncation error for Euler's method, we use Taylor's formula with remainder L a Construct y(x) possesses k + 1 derivatives that are continuous on an op 2. It'r a containing *a* and *x*, then

$$y(x_{n+1}) = y_n a_1 + y'(a) \frac{x-a}{a!} = y_n b_1 y'^{(k)}(a) \frac{(x-a)^k}{k!} + y^{(k+1)}(c) \frac{(x-a)^{k+1}}{(k+1)!},$$
where *c* is some wint between *a* and *x*. Setting *k* = 1, *a* = *x_n*, and *x* = *x_{n+1} = *x_n* + *h*,
we get
$$y(x_{n+1}) = y(x_n) + y'(x_n) \frac{h}{1!} + y''(c) \frac{h^2}{2!}$$
or
$$y(x_{n+1}) = y_n + hf(x_n, y_n) + y''(c) \frac{h^2}{2!}.$$*

v

OI

Euler's method (1) is the last formula without the last term; hence the local truncation error in y_{n+1} is

$$f'(c) \frac{h^2}{2!}, \quad \text{where} \quad x_n < c < x_{n+1}.$$

The value of c is usually unknown (it exists theoretically), so the *exact* error cannot be calculated, but an upper bound on the absolute value of the error is $Mh^2/2!$, where $M = \max_{x_n < x < x_{n+1}} |y''(x)|.$

In discussing errors that arise from the use of numerical methods, it is helpful to use the notation $O(h^n)$. To define this concept, we let e(h) denote the error in a numerical calculation depending on h. Then e(h) is said to be of order h^n , denoted by $O(h^n)$, if there exist a constant *C* and a positive integer *n* such that $|e(h)| \leq Ch^n$ for *h* sufficiently small. Thus the local truncation error for Euler's method is $O(h^2)$. We note that, in general, if e(h) in a numerical method is of order h^n and h is halved, the new error is approximately $C(h/2)^n = Ch^n/2^n$; that is, the error is reduced by a factor of $1/2^n$.



FIGURE 9.1.1 Slope of red dashed line is the average of m_0 and m_1

 $x_1 = x_0 + h$. These equations can be readily visualized. In Figure 9.1.1, observe that $m_0 = f(x_0, y_0)$ and $m_1 = f(x_1, y_1^*)$ are slopes of the solid straight lines shown passing through the points (x_0, y_0) and (x_1, y_1^*) , respectively. By taking an average of these $f(x_0, y_0) = f(x_0, y_0) + f(x_0, y_0^*)$

slopes, that is, $m_{\text{ave}} = \frac{f(x_0, y_0) + f(x_1, y_1^*)}{2}$, we obtain the slope of the parallel

dashed skew lines. With the first step, rather than advancing along the line through (x_0, y_0) with slope $f(x_0, y_0)$ to the point with *y*-coordinate y_1^* obtained by Euler's method, we advance instead along the red dashed line through (x_0, y_0) with slope m_{ave} until we reach x_1 . It seems plausible from inspection of the figure that y_1 is an improvement over y_1^* .

In general, the improved Euler's method is an example of a **predictor-corrector method.** The value of y_{n+1}^* given by (4) predicts a value of $y(x_n)$, whereas the value of y_{n+1} defined by formula (3) corrects this estimate

EXAMPLE 2 Improved Euler's Method

Use the improved Euler's method to obtain the approximate value of y(1.5) for the solution of the initial-value problem y' = 2xy, y(1) = 1. Compare the results for h = 0.1 and h = 0.05.

SOLUTION With $x_0 = 1$, $y_0 = 1$, $f(x_n, y_n) = 2x_ny_n$, n = 0, and h = 0.1, we first compute (4):

$$y_1^* = y_0 + (0.1)(2x_0y_0) = 1 + (0.1)2(1)(1) = 1.2.$$

We use this last value in (3) along with $x_1 = 1$ n = 0 n = 1.1:

$$y_1 = y_0 + (0.1) \frac{2x_0y_0 + 2x_0^*}{2} + (0.1) \frac{2(1)(1) + 2(1.1)(1.2)}{2} = 1.232.$$

have values of the particulations for h = 0.1 and h = 0.05 are given in 3 and 9.1.4, respectively.

TABLE	Eproved Euler's	A hor a th	e _{0.1}	I I
			0/ D 1	

x_n	y_n	Actual value	Abs. error	% Rel. error		<i>x</i> _n	Уn	Actual value	Abs. error	% Rel. error
1.00	1.0000	1.0000	0.0000	0.00		1.00	1.0000	1.0000	0.0000	0.00
1.10	1.2320	1.2337	0.0017	0.14		1.05	1.1077	1.1079	0.0002	0.02
1.20	1.5479	1.5527	0.0048	0.31		1.10	1.2332	1.2337	0.0004	0.04
1.30	1.9832	1.9937	0.0106	0.53		1.15	1.3798	1.3806	0.0008	0.06
1.40	2.5908	2.6117	0.0209	0.80		1.20	1.5514	1.5527	0.0013	0.08
1.50	3.4509	3.4904	0.0394	1.13		1.25	1.7531	1.7551	0.0020	0.11
					-	1.30	1.9909	1.9937	0.0029	0.14
						1.35	2.2721	2.2762	0.0041	0.18
						1.40	2.6060	2.6117	0.0057	0.22

TABLE 9.1.4 Improved Euler's Method with h = 0.05

3.0117

3.4904

0.0079

0.0108

0.26

0.31

_

A brief word of caution is in order here. We cannot compute all the values of y_n^* first and then substitute these values into formula (3). In other words, we cannot use the data in Table 9.1.1 to help construct the values in Table 9.1.3. Why not?

3.0038

3.4795

Truncation Errors for the Improved Euler's Method The local truncation error for the improved Euler's method is $O(h^3)$. The derivation of this result is similar to the derivation of the local truncation error for Euler's method. Since the

1.45

1.50

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9.2 RUNGE-KUTTA METHODS

REVIEW MATERIAL

• Section 2.6 (see page 78)

INTRODUCTION Probably one of the more popular as well as most accurate numerical procedures used in obtaining approximate solutions to a first-order initial-value problem y' = f(x, y), $y(x_0) = y_0$ is the **fourth-order Runge-Kutta method.** As the name suggests, there are Runge-Kutta methods of different orders.

Runge-Kutta Methods Fundamentally, all Runge-Kutta methods are generalizations of the basic Euler formula (1) of Section 9.1 in that the slope function *f* is replaced by a weighted average of slopes over the interval $x_n \le x \le x_{n+1}$. That is,

$$y_{n+1} = y_n + h \left(w_1 k_1 + w_2 k_2 + \dots + w_m k_m \right).$$
(1)

Here the weights w_i , i = 1, 2, ..., m, are constants that generally satisfy $w_1 + w_2 + \cdots + w_m = 1$, and each k_i , i = 1, 2, ..., m, is the function f evaluated at a selected point (x, y) for which $x_n \le x \le x_{n+1}$. We shall see that the k_i are defined recursively. The number m is called the **order** of the method. Observe that by taking m = 1, $w_1 = 1$, and $k_1 = f(x_n, y_n)$, we get the diministration formula $y_{n+1} = y_n + hf(x_n, y_n)$. Hence Euler's method is said to be a **first-order Runge-Kutta method**.

The average in (1) is no formed willy-nilly, but parameters are chosen so that (1) agrees with a 1 r polynomial of average m. As we saw in the preceding section, if are notion y(x) possesser k + i or a varives that are continuous on an open interval containing a and x then y can write

Preview
$$page(x) = y(a) + y'(a) \frac{x-a}{1!} + y''(a) \frac{(x-a)^2}{2!} + \dots + y^{(k+1)}(c) \frac{(x-a)^{k+1}}{(k+1)!},$$

where *c* is some number between *a* and *x*. If we replace *a* by x_n and *x* by $x_{n+1} = x_n + h$, then the foregoing formula becomes

$$y(x_{n+1}) = y(x_n + h) = y(x_n) + hy'(x_n) + \frac{h^2}{2!}y''(x_n) + \cdots + \frac{h^{k+1}}{(k+1)!}y^{(k+1)}(c),$$

where *c* is now some number between x_n and x_{n+1} . When y(x) is a solution of y' = f(x, y) in the case k = 1 and the remainder $\frac{1}{2}h^2y''(c)$ is small, we see that a Taylor polynomial $y(x_{n+1}) = y(x_n) + hy'(x_n)$ of degree one agrees with the approximation formula of Euler's method

$$y_{n+1} = y_n + hy'_n = y_n + hf(x_n, y_n).$$

A Second-Order Runge-Kutta Method To further illustrate (1), we consider now a **second-order Runge-Kutta procedure.** This consists of finding constants or parameters w_1 , w_2 , α , and β so that the formula

$$y_{n+1} = y_n + h(w_1k_1 + w_2k_2),$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + h, y_n + \beta h k_1),$$
(2)

where

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It can be proved that a determinant det **A** can be expanded by cofactors using any row or column. If det **A** has a row (or a column) containing many zero entries, then wisdom dictates that we expand the determinant by that row (or column).

DEFINITION II.7 Transpose of a Matrix

The **transpose** of the $m \times n$ matrix (1) is the $n \times m$ matrix \mathbf{A}^T given by

 $\mathbf{A}^{T} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}.$

In other words, the rows of a matrix \mathbf{A} become the columns of its transpose \mathbf{A}^{T} .

EXAMPLE 7 Transpose of a Matrix (a) The transpose of $\mathbf{A} = \begin{pmatrix} 3 & 6 & 2 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{pmatrix}$ is $\mathbf{A}^T = \begin{pmatrix} 3 & 2 & -1 \\ 6 & 5 & 2 \\ 2 & 1 & 4 \end{pmatrix}$. (b) If $\mathbf{X} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$, then $\mathbf{X}^T = (5 \ 0 \ 3)$ of \mathbf{U} . **EXAMPLE 2 DEFINITION II.8 INFORMATION III.8 INFORMATION II.8 INFORMATION III.8 INFORMATION II**

DEFINITION II.9 Nonsingular/Singular Matrices

Let A be an $n \times n$ matrix. If det $A \neq 0$, then A is said to be **nonsingular**. If det A = 0, then A is said to be **singular**.

The following theorem gives a necessary and sufficient condition for a square matrix to have a multiplicative inverse.

THEOREM II.1 Nonsingularity Implies A Has an Inverse

An $n \times n$ matrix **A** has a multiplicative inverse \mathbf{A}^{-1} if and only if **A** is nonsingular.

The following theorem gives one way of finding the multiplicative inverse for a nonsingular matrix.
and
$$\int_0^t \mathbf{X}(s) \, ds = \begin{pmatrix} \int_0^t \sin 2s \, ds \\ \int_0^t e^{3s} \, ds \\ \int_0^t (8s - 1) \, ds \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \cos 2t + \frac{1}{2} \\ \frac{1}{3} e^{3t} - \frac{1}{3} \\ 4t^2 - t \end{pmatrix}.$$

II.2 GAUSSIAN AND GAUSS-JORDAN ELIMINATION

Matrices are an invaluable aid in solving algebraic systems of n linear equations in n variables or unknowns,

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n.$$

(5)

If **A** denotes the matrix of coefficients in (5), we know that Cramer's rule could be used to solve the system whenever det $\mathbf{A} \neq 0$. However, that rule requires a herculean effort if **A** is larger than 3×3 . The procedure that we shall now consider has the distinct advantage of being not only an efficient way of handling large systems, but also a means of solving consistent systems (5) in which det $\mathbf{A} = 0$ and a means of solving *m* linear equations in *n* unknowns.



Elementary Row Operations Recall from algebra that we can transform an algebraic system of equations into an equivalent system (that is, one having the same solution) by multiplying an equation by a nonzero constant, interchanging the positions of any two equations in a system, and adding a nonzero constant multiple of an equation to another equation. These operations on equations in a system are, in turn, equivalent to **elementary row operations** on an augmented matrix:

- (*i*) Multiply a row by a nonzero constant.
- (*ii*) Interchange any two rows.
- (iii) Add a nonzero constant multiple of one row to any other row.

Elimination Methods To solve a system such as (5) using an augmented matrix, we use either **Gaussian elimination** or the **Gauss-Jordan elimination method**. In the former method, we carry out a succession of elementary row operations until we arrive at an augmented matrix in **row-echelon form**:

- (*i*) The first nonzero entry in a nonzero row is 1
- (*ii*) In consecutive nonzero rows the first entry 1 in the lower row appears t the right of the first 1 in the higher ro .
- (*iii*) Rows consisting of all 0's are at the bottom of the matrix.

In Problems 55 and 56 show that the given matrix has complex eigenvalues. Find the eigenvectors of the matrix.

55.
$$\begin{pmatrix} -1 & 2 \\ -5 & 1 \end{pmatrix}$$
 56. $\begin{pmatrix} 2 & -1 & 0 \\ 5 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix}$

Miscellaneous Problems

57. If A(t) is a 2 × 2 matrix of differentiable functions and X(t) is a 2 × 1 column matrix of differentiable functions, prove the product rule

$$\frac{d}{dt} \left[\mathbf{A}(t) \mathbf{X}(t) \right] = \mathbf{A}(t) \mathbf{X}'(t) + \mathbf{A}'(t) \mathbf{X}(t).$$

58. Derive formula (3). [*Hint*: Find a matrix

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

for which AB = I. Solve for b_{11} , b_{12} , b_{21} , and b_{22} . Then show that BA = I.]

- **59.** If A is nonsingular and AB = AC, show that B = C.
- **60.** If **A** and **B** are nonsingular, show that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
- **61.** Let **A** and **B** be $n \times n$ matrices. In general, is

 $(A + B)^2 = A^2 + 2AB + B^2?$

- **62.** A square matrix **A** is said to be a **diagonal matrix** if all its entries off the main diagonal are zero—that is, $a_{ij} = 0, i \neq j$. The entries a_{ii} on the main diagonal may or may not be zero. The multiplicative identity matrix **I** is an example of a diagonal matrix.
 - (a) Find the inverse of the 2×2 diagonal matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & 0\\ 0 & a_{22} \end{pmatrix}$$

when $a_{11} \neq 0, a_{22} \neq 0$.

- (b) Find the inverse of a 3×3 diagonal matrix A whose main diagonal entries a_{ii} are all nonzero.
- (c) In general, what is the inverse of an $n \times n$ diagonal matrix **A** whose main diagonal entries a_{ii} are all nonzero?
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4(0)

25.
$$2y - 2x + \sin 2(x + y) = c$$

27. $4(y - 2x + 3) = (x + c)^2$
29. $-\cot(x + y) + \csc(x + y) = x + \sqrt{2} - 1$
35. (b) $y = \frac{2}{x} + \left(-\frac{1}{4}x + cx^{-3}\right)^{-1}$

EXERCISES 2.6 (PAGE 79)

1. $y_2 = 2.9800, y_4 = 3.1151$ **3.** $y_{10} = 2.5937$, $y_{20} = 2.6533$; $y = e^x$ **5.** $y_5 = 0.4198$, $y_{10} = 0.4124$ **7.** $y_5 = 0.5639$, $y_{10} = 0.5565$ **9.** $y_5 = 1.2194$, $y_{10} = 1.2696$ **13.** Euler: $y_{10} = 3.8191$, $y_{20} = 5.9363$ RK4: $y_{10} = 42.9931$, $y_{20} = 84.0132$

CHAPTER 2 IN REVIEW (PAGE 80)

1. -A/k, a repeller for k > 0, an attractor for k < 03. true d^3v

5.
$$\frac{d^2 y}{dx^3} = x \sin y$$

- 7. true
- **9.** $y = c_1 e^{e^x}$

11.
$$\frac{dy}{dx} + (\sin x)y = x$$

13. $\frac{dy}{dx} = (y - 1)^2 (y - 3)^2$

15. semi-stable for *n* even and unstrip semi-stable for *n* even at a sy n for *n* odd.

EXERCISES 3.1 (PAGE 90)

- **1.** 7.9 yr; 10 yr
- 3. 760; approximately 11 persons/yr
- 5. 11 h
- 7. 136.5 h
- **9.** $I(15) = 0.00098I_0$ or approximately 0.1% of I_0
- 11. 15,600 years
- **13.** $T(1) = 36.67^{\circ}$ F; approximately 3.06 min
- 15. approximately 82.1 s; approximately 145.7 s
- **17.** 390°
- **19.** about 1.6 hours prior to the discovery of the body

21. $A(t) = 200 - 170e^{-t/50}$

23.
$$A(t) = 1000 - 1000e^{-t/100}$$

25. $A(t) = 1000 - 10t - \frac{1}{10}(100 - t)^2$; 100 min
27. 64.38 lb
29. $i(t) = \frac{3}{5} = \frac{5}{5}e^{-50t}$; $i \to \frac{3}{5}$ as $t \to \infty$
31. $q(t) = \frac{1}{100} - \frac{1}{100}e^{-50t}$; $i(t) = \frac{1}{2}e^{-50t}$
33. $i(t) = \begin{cases} 60 - 60e^{-t/10}, & 0 \le t \le 20\\ 60(e^2 - 1)e^{-t/10}, & t > 20 \end{cases}$
35. (a) $v(t) = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-kt/m}$
(b) $v \to \frac{mg}{k}$ as $t \to \infty$
(c) $s(t) = \frac{mg}{k}t - \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)e^{-kt/m}$
 $+ \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)$
39. (a) $v(t) = \frac{g}{4k}\left(\frac{k}{t} + t + r_0\right) - \frac{gr_0}{4k}\left(\frac{r_0}{k-t} + r_0\right)^3$
(c) $33\frac{1}{3}$ seconds
41. (a) $P(t) = 0e^{\frac{d}{2}\sqrt{1-t}} + \frac{d}{4}(t) = 0e^{\frac{d}{2}\sqrt{1-t}} +$

(b) 576 $\sqrt{10}$ s or 30.36 min

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13. (a) approximately 858.65 s or 14.31 min **(b)** 243 s or 4.05 min

В

EXERCISES 4.2 (PAGE 131)

1. $y_2 = xe^{2x}$	3. $y_2 = \sin 4x$
5. $y_2 = \sinh x$	7. $y_2 = xe^{2x/3}$
9. $y_2 = x^4 \ln x $	11. $y_2 = 1$
13. $y_2 = x \cos(\ln x)$	15. $y_2 = x^2 + x + 2$
17. $y_2 = e^{2x}, y_p = -\frac{1}{2}$	19. $y_2 = e^{2x}, y_p = \frac{5}{2}e^{3x}$

EXERCISES 4.3 (PAGE 137)

3. $y = c_1 e^{3x} + c_2 e^{-2x}$ **1.** $y = c_1 + c_2 e^{-x/4}$ 5. $y = c_1 e^{-4x} + c_2 x e^{-4x}$ 7. $y = c_1 e^{2x/3} + c_2 e^{-x/4}$ 9. $y = c_1 \cos 3x + c_2 \sin 3x$ 11. $y = e^{2x}(c_1 \cos x + c_2 \sin x)$ **13.** $y = e^{-x/3} \left(c_1 \cos \frac{1}{3} \sqrt{2} x + c_2 \sin \frac{1}{3} \sqrt{2} x \right)$ **15.** $y = c_1 + c_2 e^{-x} + c_3 e^{5x}$ **17.** $y = c_1 e^{-x} + c_2 e^{3x} + c_3 x e^{3x}$ **19.** $u = c_1 e^t + e^{-t} (c_2 \cos t + c_3 \sin t)$ **21.** $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$ **23.** $y = c_1 + c_2 x + e^{-x/2} \left(c_3 \cos \frac{1}{2} \sqrt{3} x + c_4 \sin \frac{1}{2} \sqrt{3} x \right)$ **25.** $y = c_1 \cos \frac{1}{2} \sqrt{3} x + c_2 \sin \frac{1}{2} \sqrt{3} x$ $+ c_3 x \cos \frac{1}{2} \sqrt{3} x + c_4 x \sin \frac{1}{2} \sqrt{3} x$ **27.** $u = c_1 e^r + c_2 r e^r + c_3 e^{-r} + c_4 r e^{-r} + c_5 e^{-5r}$ **29.** $y = 2\cos 4x - \frac{1}{2}\sin 4x$ **31.** $y = -\frac{1}{3}e^{-(t-1)} + \frac{1}{3}e^{5(t-1)}$ **33.** y = 0**35.** $y = \frac{5}{36} - \frac{5}{36}e^{-6x} + \frac{1}{6}xe^{-6x}$ **37.** $y = e^{5x} - xe^{5x}$ **39.** y = 0**49.** y'' - 6y' + 5y = 0**51.** y'' - 2y' = 0**55.** y'' + 2y' + 2y = 0**53.** y'' + 9y = 0**57.** y''' - 8y'' = 0

EXERCISES 4.4 (PAGE 147)

1. $y = c_1 e^{-x} + c_2 e^{-2x} + 3$ 3. $y = c_1 e^{5x} + c_2 x e^{5x} + \frac{6}{5} x + \frac{3}{5}$ 5. $y = c_1 e^{-2x} + c_2 x e^{-2x} + x^2 - 4x + \frac{7}{2}$ 7. $y = c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x + (-4x^2 + 4x - \frac{4}{3})e^{3x}$ 9. $y = c_1 + c_2 e^x + 3x$ 11. $y = c_1 e^{x/2} + c_2 x e^{x/2} + 12 + \frac{1}{2}x^2 e^{x/2}$ 13. $y = c_1 \cos 2x + c_2 \sin 2x - \frac{3}{4}x \cos 2x$ 15. $y = c_1 \cos x + c_2 \sin x - \frac{1}{2}x^2 \cos x + \frac{1}{2}x \sin x$ 17. $y = c_1 e^x \cos 2x + c_2 e^x \sin 2x + \frac{1}{4}x e^x \sin 2x$ 19. $y = c_1 e^{-x} + c_2 x e^{-x} - \frac{1}{2} \cos x + \frac{12}{25} \sin 2x - \frac{9}{25} \cos 2x$

21.
$$y = c_1 + c_2 x + c_3 e^{6x} - \frac{1}{4} x^2 - \frac{6}{37} \cos x + \frac{1}{37} \sin x$$

23. $y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x - x - 3 - \frac{2}{3} x^3 e^x$
25. $y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x + x^2 - 2x - 3$
27. $y = \sqrt{2} \sin 2x - \frac{1}{2}$
29. $y = -200 + 200 e^{-x/5} - 3x^2 + 30x$
31. $y = -10e^{-2x} \cos x + 9e^{-2x} \sin x + 7e^{-4x}$
33. $x = \frac{F_0}{2\omega^2} \sin \omega t - \frac{F_0}{2\omega} t \cos \omega t$
35. $y = 11 - 11e^x + 9xe^x + 2x - 12x^2e^x + \frac{1}{2}e^{5x}$
37. $y = 6 \cos x - 6(\cot 1) \sin x + x^2 - 1$
39. $y = \frac{-4 \sin \sqrt{3}x}{\sin \sqrt{3} + \sqrt{3} \cos \sqrt{3}} + 2x$
41. $y = \begin{cases} \cos 2x + \frac{5}{6} \sin 2x + \frac{1}{3} \sin x, & 0 \le x \le \pi/2 \\ \frac{2}{3} \cos 2x + \frac{5}{6} \sin 2x, & x > \pi/2 \end{cases}$

EXERCISES 4.5 (PAGE 155)

1. $(3D - 2)(3D + 2)y = \sin x$ 3. (D-6)(D+2)y = x - 65. $D(D+5)^2y = e^x$ 5. D(D + 5) y - e7. $(D - 1)(D - 2)(D + 5)y = xe^{-x}$ 9. D(D + 2)D -4)v = 4**17.** D(D-2)**21.** $D^3(D^2 + 16)$ **25.** $D(D^2 - 2D + 5)$ **29.** e^{6x} , $e^{-3x/2}$ **33.** 1, e^{5x} , xe^{5x} **35.** $y = c_1 e^{-3x} + c_2 e^{3x} - 6$ **37.** $y = c_1 + c_2 e^{-x} + 3x$ **39.** $y = c_1 e^{-2x} + c_2 x e^{-2x} + \frac{1}{2}x + 1$ **41.** $y = c_1 + c_2 x + c_3 e^{-x} + \frac{2}{3} x^4 - \frac{8}{3} x^3 + 8x^2$ **43.** $y = c_1 e^{-3x} + c_2 e^{4x} + \frac{1}{7} x e^{4x}$ **45.** $y = c_1 e^{-x} + c_2 e^{3x} - e^x + 3$ **47.** $y = c_1 \cos 5x + c_2 \sin 5x + \frac{1}{4} \sin x$ **49.** $y = c_1 e^{-3x} + c_2 x e^{-3x} - \frac{1}{49} x e^{4x} + \frac{2}{343} e^{4x}$ **51.** $y = c_1 e^{-x} + c_2 e^x + \frac{1}{6} x^3 e^x - \frac{1}{4} x^2 e^x + \frac{1}{4} x e^x - 5$ 53. $y = e^{x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{3}e^{x} \sin x$ **55.** $y = c_1 \cos 5x + c_2 \sin 5x - 2x \cos 5x$ **57.** $y = e^{-x/2} \left(c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right)$ $+\sin x + 2\cos x - x\cos x$ **59.** $y = c_1 + c_2 x + c_3 e^{-8x} + \frac{11}{256} x^2 + \frac{7}{32} x^3 - \frac{1}{16} x^4$ **61.** $y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + \frac{1}{6} x^3 e^x + x - 13$ **63.** $y = c_1 + c_2 x + c_3 e^x + c_4 x e^x + \frac{1}{2} x^2 e^x + \frac{1}{2} x^2$ **65.** $y = \frac{5}{8}e^{-8x} + \frac{5}{8}e^{8x} - \frac{1}{4}$ **67.** $y = -\frac{41}{125} + \frac{41}{125}e^{5x} - \frac{1}{10}x^2 + \frac{9}{25}x$ **69.** $y = -\pi \cos x - \frac{11}{3} \sin x - \frac{8}{3} \cos 2x + 2x \cos x$ 71. $y = 2e^{2x}\cos 2x - \frac{3}{64}e^{2x}\sin 2x + \frac{1}{8}x^3 + \frac{3}{16}x^2 + \frac{3}{32}x$

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EXERCISES 4.6 (PAGE 161)

1. $y = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \ln |\cos x|$ 3. $y = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x$ 5. $y = c_1 \cos x + c_2 \sin x + \frac{1}{2} - \frac{1}{6} \cos 2x$ 7. $y = c_1 e^x + c_2 e^{-x} + \frac{1}{2}x \sinh x$ **9.** $y = c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{4} \left(e^{2x} \ln |x| - e^{-2x} \int_x^x \frac{e^{4t}}{t} dt \right),$ $x_0 > 0$ **11.** $y = c_1 e^{-x} + c_2 e^{-2x} + (e^{-x} + e^{-2x}) \ln(1 + e^x)$ **13.** $y = c_1 e^{-2x} + c_2 e^{-x} - e^{-2x} \sin e^x$ **15.** $y = c_1 e^{-t} + c_2 t e^{-t} + \frac{1}{2} t^2 e^{-t} \ln t - \frac{3}{4} t^2 e^{-t}$ **17.** $y = c_1 e^x \sin x + c_2 e^x \cos x + \frac{1}{3} x e^x \sin x$ $+\frac{1}{3}e^x \cos x \ln |\cos x|$ **19.** $y = \frac{1}{4}e^{-x/2} + \frac{3}{4}e^{x/2} + \frac{1}{8}x^2e^{x/2} - \frac{1}{4}xe^{x/2}$ **21.** $y = \frac{4}{9}e^{-4x} + \frac{25}{24}e^{2x} - \frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x}$ **23.** $y = c_1 x^{-1/2} \cos x + c_2 x^{-1/2} \sin x + x^{-1/2}$ **25.** $y = c_1 + c_2 \cos x + c_3 \sin x - \ln |\cos x|$ $-\sin x \ln |\sec x + \tan x|$ **27.** $y = c_1 e^x + c_2 e^{-x} + c_3 e^{2x} + \frac{1}{30} e^{4x}$ EXERCISES 4.7 (PAGE 168) 1. $y = c_1 x^{-1} + c_2 x^2$ from Not nx) age 464 c 3. $y = c_1 + c_2 \ln x$ 5. $y = c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)$ 7. $y = c_1 x^{(2-\sqrt{6})} + c_2 x^{(2+\sqrt{6})}$ **9.** $y = c_1 \cos\left(\frac{1}{5} \ln x\right)$ $\frac{1}{5} \ln x$ $c_{12} - c_{2}x^{-2} \ln x$ **11.** y **13.** $y = x^{-1/2} \left[c_1 \cos\left(\frac{1}{6}\sqrt{3}\ln x\right) + c_2 \sin\left(\frac{1}{6}\sqrt{3}\ln x\right) \right]$ **15.** $y = c_1 x^3 + c_2 \cos(\sqrt{2} \ln x) + c_3 \sin(\sqrt{2} \ln x)$ **17.** $y = c_1 + c_2 x + c_3 x^2 + c_4 x^{-3}$ **19.** $y = c_1 + c_2 x^5 + \frac{1}{5} x^5 \ln x$ **21.** $y = c_1 x + c_2 x \ln x + x (\ln x)^2$ **23.** $y = c_1 x^{-1} + c_2 x - \ln x$ **25.** $y = 2 - 2x^{-2}$ **27.** $y = \cos(\ln x) + 2\sin(\ln x)$ **29.** $y = \frac{3}{4} - \ln x + \frac{1}{4}x^2$ **31.** $y = c_1 x^{-10} + c_2 x^2$ **33.** $y = c_1 x^{-1} + c_2 x^{-8} + \frac{1}{30} x^2$ **35.** $y = x^2 [c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)] + \frac{4}{13} + \frac{3}{10}x$ **37.** $y = 2(-x)^{1/2} - 5(-x)^{1/2} \ln(-x), x < 0$ **39.** $y = c_1(x + 3)^2 + c_2(x + 3)^7$ **41.** $y = c_1 \cos[\ln(x+2)] + c_2 \sin[\ln(x+2)]$

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EXERCISES 4.8 (PAGE 179)

1.
$$y_p(x) = \frac{1}{4} \int_{x_0}^x \sinh 4(x-t)f(t)dt$$

3. $y_p(x) = \int_{x_0}^x (x-t)e^{-(x-t)}f(t)dt$
5. $y_p(x) = \frac{1}{3} \int_{x_0}^x \sin 3(x-t)f(t)dt$
7. $y = c_1e^{-4x} + c_2e^{4x} + \frac{1}{4} \int_{x_0}^x \sinh 4(x-t)te^{-2t}dt$
9. $y = c_1e^{-x} + c_2xe^{-x} + \int_{x_0}^x (x-t)e^{-(x-t)}e^{-t}dt$
11. $y = c_1\cos 3x + c_2\sin 3x + \frac{1}{3} \int_{x_0}^x \sin 3(x-t)(t+\sin t)dt$
13. $y_p(x) = \frac{1}{4}xe^{2x} - \frac{1}{16}e^{2x} + \frac{1}{16}e^{-2x}$
15. $y_p(x) = \frac{1}{2}x^2e^{5x}$
17. $y_p(x) = -\cos x + \frac{\pi}{2}\sin x - x\sin x - \cos x\ln|\sin x|$
19. $y = \frac{25}{16}e^{-2x} - \frac{9}{16}e^{2x} + \frac{1}{4}xe^{2x}$
21. $y = -e^{5x} + 6xe^{5x} + \frac{1}{2}x^2e^{5x}$
23. $y = -x\sin x - \cos x\ln|\sin x|$
25. $y = (\cos 1 - 2)e^{-x} + \frac{1}{36} - \frac{1}{6}\ln x$
27. $u = \sqrt{16x^3} - \frac{1}{16}x^{-2} + \frac{1}{36} - \frac{1}{6}\ln x$
28. $y(x) = 5e^x + 3e^{-x} + y_p(x),$
where $y_p(x) = \begin{cases} 1 - \cosh x, x < 0 \\ -1 + \cosh x, x & 0 \end{cases}$
33. $y = \cos x - \sin x + y_p(x),$
where $y_p(x) = \begin{cases} 0, & x < 0 \\ 10 - 10\cos x, & 0 \le x \le 3\pi \\ -20\cos x, & x > 3\pi \end{cases}$
35. $y_p(x) = (x-1)\int_0^x tf(t)dt + x\int_x^1(t-1)f(t)dt$
37. $y_p(x) = \frac{1}{2}x^2 - \frac{1}{2}x$
39. $y_p(x) = \frac{\sin(x-1)}{\sin 1} - \frac{\sin x}{\sin 1} + 1$
41. $y_p(x) = -e^x\cos x - e^x\sin x + e^x$
43. $y_p(x) = \frac{1}{2}(\ln x)^2 + \frac{1}{2}\ln x$

EXERCISES 4.9 (PAGE 184)

1. $x = c_1 e^t + c_2 t e^t$ $y = (c_1 - c_2) e^t + c_2 t e^t$ 3. $x = c_1 \cos t + c_2 \sin t + t + 1$ $y = c_1 \sin t - c_2 \cos t + t - 1$ ANS-16 • ANSWERS FOR SELECTED ODD-NUMBERED PROBLEMS

EXERCISES 8.4 (PAGE 359)

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EXERCISES 8.4 (PAGE 359)
1.
$$e^{At} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix}$$
; $e^{-At} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$
3. $e^{At} = \begin{pmatrix} t+1 & t & t \\ t & t+1 & t \\ -2t & -2t & -2t + 1 \end{pmatrix}$
5. $X = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}$
11. $X = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t}$
13. $X = c_1 \begin{pmatrix} t+1 \\ -2t \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 \\ -2t \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 \\ -2t \end{pmatrix} + c_2 \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} e^{t}$
14. $X = c_1 \begin{pmatrix} 0 \\ 0 \\ e^{2t} \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t}$
15. $(b) X = c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t}$
16. $(b) X = c_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t}$
17. $X = c_1 \begin{pmatrix} e^{t} + 1 \\ t \\ -2t \end{pmatrix} - 4 \begin{pmatrix} t \\ t + 1 \\ -2t \end{pmatrix} + 6 \begin{pmatrix} t \\ t \\ -2t \end{pmatrix}$
18. $X = c_1 \begin{pmatrix} 2e^{t} - 2e^{t} - 2e^{t} + e^{-2t} \end{pmatrix}$
19. $X = c_1 \begin{pmatrix} 2e^{t} - 2e^{t} - 2e^{t} + e^{-2t} \end{pmatrix} + c_2 \begin{pmatrix} 1e^{t} - 1e^{t} + 1e^{t} - 1e^{t} + 1e^$

1.
$$k = \frac{1}{3}$$

5. $\mathbf{X} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t \right]$
7. $\mathbf{X} = c_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^t + c_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^t$

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(b) $y''(c)\frac{h^2}{2} \le 19(0.1)^2(1) = 0.19$

(d) Error with h = 0.1 is 0.2325. Error with h = 0.05

(c) If h = 0.1, $y_5 = 1.8207$. If h = 0.05, $y_{10} = 1.9424$.

is 0.1109.

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