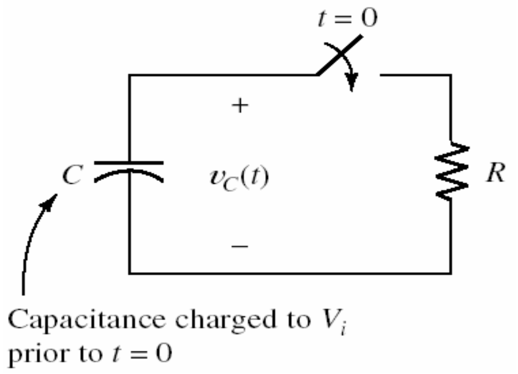
Discharge of a Capacitance Through a Resistance



$$RCKse^{st} + Ke^{st} = 0$$

$$(RCs + 1)Ke^{st} = 0$$

$$RCs + 1 = 0$$

$$\boxed{s = \frac{-1}{RC}}$$

$$v_C(t) = Ke^{-t/RC}$$

Voltage immediately after the switch closes

$$v_C(0+) = V_i$$

$$v_C(0^+) = V_i = Ke^0 = K$$

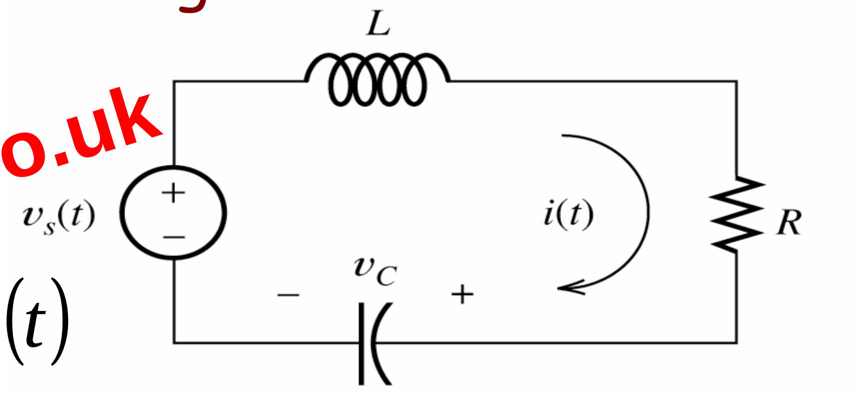
Since voltage across the capacitor cannot change instantaneously
(current would be infinite)

Thus,

$$v_C(t) = V_i e^{-t/RC}$$

Second-Order Circuits

Circuits with two energy-storage elements



(a) Electrical circuit

Write KVL:

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(t)\,dt+v_C(0)=v_s(t)$$

$$L\frac{d^2i(t)}{dt^2}+R\frac{di(t)}{dt}+\frac{1}{C}i(t)=\frac{dv_s(t)}{dt}$$

$$\frac{d^2i(t)}{dt^2}+\frac{R}{L}\frac{di(t)}{dt}+\frac{1}{LC}i(t)=\frac{1}{L}\frac{dv_s(t)}{dt}$$

Define:

$\alpha = \frac{R}{2L}$ (Damping coefficient)

$\omega_0 = \frac{1}{\sqrt{LC}}$ (Undamped resonant frequency)

$f(t)=\frac{1}{L}\frac{dv_S(t)}{dt}$ (forcing function)

$$\boxed{\frac{d^2i(t)}{dt^2}+2\alpha\frac{di(t)}{dt}+\omega_0^2 i(t)=f(t)}$$

Second-order DE

Solution to the DE:

$$\boxed{(s^2 + 2\alpha s + \omega_0^2) = 0}$$

The roots of the characteristic equation are:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

Define the damping ratio as: $\zeta = \dfrac{\alpha}{\omega_0}$

The form of the complementary solution depends on the value of the damping ratio

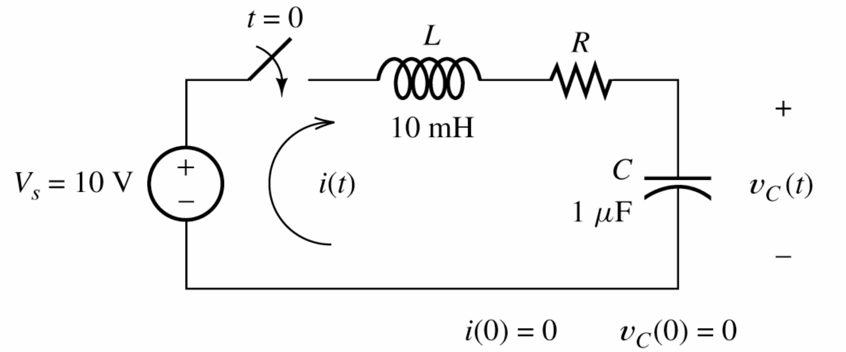
		Roots (s1, s2)
$\zeta > 1 \;\; (\alpha > \omega_0) \longrightarrow$	Overdamped	Real and distinct
$\zeta = 1 \;\; (\alpha = \omega_0) \longrightarrow$	Critically damped	Real and equal
$\zeta < 1 \;\; (\alpha < \omega_0) \longrightarrow$	Underdamped	Complex

Example (cont.)

$$i(t) = C\frac{dv_c(t)}{dt}$$

We need to find the $v_c(t)$ for $R = 300, 200, 100\ \Omega$

$$L\frac{di(t)}{dt} + Ri(t) + v_c(t) = V_s$$



$$LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t) = V_s$$

$$\frac{d^2v_c(t)}{dt^2} + \frac{R}{L}\frac{dv_c(t)}{dt} + \frac{1}{LC}v_c(t) = \frac{V_s}{LC}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 10000$$

$$\alpha = \frac{R}{2L} = \begin{cases} 15000 & \text{if} \quad R = 300 \\ 10000 & \text{if} \quad R = 200 \\ 5000 & \text{if} \quad R = 100 \end{cases}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$ (Undamped resonant frequency)

$$\alpha = \frac{R}{2L}$$ (Damping coefficient)

$$f(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$$ (forcing function)

Comparing the three cases