These structure equations already have global implications. For example, suppose that Σ were compact. Then it would necessarily be geodesically complete, but the structure equations above show that, along any integral curve of $\mathbf{W}$, the function S satisfies an equation of the form $S'' - S = 0$, where the prime denoted differentiation with respect to the flow parameter along the integral curve. However, the only solution of this equation which is bounded in both directions is the zero solution. Since Σ is supposed to be compact, S must be bounded on Σ and hence on every integral curve. Of course this implies that S (and hence C) must vanish identically, so that the structure is locally Riemannian. In particular any Finsler structure on a compact surface M which satisfies $K \equiv -1$ (or any negative constant, for that matter) must be a Riemannian metric[11] , a result to be found in [Ak].

Note that $J = -S^2 + C^2$ is constant along the geodesic curves, i.e., the integral curves of $\mathbf{W}$, and so is a first integral of the geodesic equations. Thus, in the non-Riemannian case, the geodesic flow must be completely integrable, again, a great contrast with the Riemannian case when $K = -1$.

6.1. Canonical structures on the geodesic space. Consider the 1-form $S\,\theta - C\,\eta$, the 2-form $\eta{\wedge}\theta$, and the quadratic form $\eta^2 - \theta^2$. A computation from the structure equations yields

$$\mathfrak{L}_{\mathbf{W}}\big(S\,\theta - C\,\eta\big) = \mathfrak{L}_{\mathbf{W}}\big(\eta\wedge\theta\big) = \mathfrak{L}_{\mathbf{W}}\big(\eta^2 - \theta^2\big) = 0,$$

so that all of these quantities are invariant under the geodesic flow.

If one assumes that the generalized Finsler structure is geodesically amenable, with geodesic projection $\ell : \Sigma \to \Lambda$, then it follows that there exist on Λ a 1-form φ so that $\ell^*\varphi = -S\,\theta + C\,\eta$, a 2-form dA so that $\ell^*(dA) = \eta{\wedge}\theta$; and a Lorentzian quadratic form g so that $\ell^*g = \eta^2 - \theta^2$.

Using the same sort of immersion of Σ into the 'orthonormal frame bundle of g as I did in the $K = 1$ case, I can identify η and θ as the canonical forms on the Lorentzian orthonormal frame bundle of g and, due to the equations

$$\begin{aligned} d\eta &= (\omega + S\,\theta - C\,\eta)\wedge\theta \\ d\theta &= (\omega + S\,\theta - C\,\eta)\wedge\eta \end{aligned}$$

one sees that $\psi = (\omega{+}S\,\theta{-}C\,\eta)$ can be thought of as the Levi-Civita connection of this pseudo-metric. The curvature R of this metric is then defined by $d\psi = R\,\eta{\wedge}\theta$ and is well-defined on Λ. Then, just as before, one derives

$$d\varphi = (1 - R)\,dA.$$

as the equation relating the 1-form φ with the oriented Lorentzian structure defined by g and the choice of oriented area form dA.

Conversely, starting with an oriented surface Λ endowed with a Lorentzian metric g of curvature R and area form dA and a 1-form φ which satisfies the

[11]Note that this result definitely does not hold for Finsler structures on a compact surface if one merely assumes that K is bounded above by a negative constant