Preface

In the present venture we present a few important aspects of Ordinary Differential equations in the form of lectures. The material is more or less dictated by the syllabus suggested by NPTEL program (courtesy MHRD, Government of India). It is only a modest attempt to gather appropriate material to cover about 39 odd lectures. While presenting the text on ordinary differential equations, we have constantly kept in mind about the readers who may not have contact hours as well as those who wish to use the text in the form lectures, hence the material is presented in the form of lectures rather than as chapters of a book.

In all there are 39 lectures. More or less a theme is selected for each lecture. A few problems are posed at the end of each lecture either for illustration or to cover a missed elements of the theme. The notes is divided into 5 modules . Module 1 dealswith existence and uniqueness of solutions for Initial Value Problems(IVP) while Module 2 deals with the structure of solutions of Linear Equations Of Higher Orders. The Study of Systems Of Linear Differential equations is the content of Module 3. Module 4 is an elementary introduction to Theory Of Oscillations and Two Point Boundary Value Problems. The notes ends with Module 5 wherein w have a brief introduction to the Asymptotic Behavior and Stability Theory. Elementary Real Analysis, Linear Algebra is a prerequisite.

I am thankful to NPTEL for giving me financial support and for an opportunity to design the web course on Ordinary Differential Equations. I am thankful to Mr. Ravikant Saini (IITK) for the patient typing of the manuscript in Tex and Dr. Rasmita Kar (PDF at TIFR, Bangalore) for the timely help to edit the manuscript. I am also thankful by the both referees for the useful and constructive comments which has improved the petition of Raghavendra V Department O Mathematics and Statistics India Institute Of Technology, Kanpur India Sentersh the material. I welcome useful suggestions from reader /users which er and may improve the lecture notes.

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Preview from Notesale.co.uk Page 4 of 157 implies the inequality

$$f(t) \le h(t) + \int_{t_0}^t g(s)h(s) \exp\left(\int_{t_0}^s g(u)du\right) ds, \quad t \ge t_0.$$

{Hint: Let $z(t) = \int_{t_0}^t g(s)f(s)ds$. Then,

$$z'(t) = g(t)f(t) \le g(t)[h(t) + z(t)].$$

Hence,

$$z'(t) - g(t)z(t) \le g(t)h(t).$$

Multiply by $\exp(-\int_{t_0}^t g(s)ds)$ on either side of this inequality and integrate over $[t_0, t]$.



Exercise : Prove that \hat{x} is a solution of (1.5) existing on $h_1 < t \le h_2$.

Now consider a rectangle around $P:(h_2, x(h_2 - 0))$ lying inside D. Consider a solution of (1.5) through P. As before, by Picard's theorem there exists a solution y through the point P on an interval

$$h_2 - \alpha \le t \le h_2 + \alpha, \ \alpha > 0$$
 and with $h_2 - \alpha \ge h_1$.

Now define z by

$$z(t) = \hat{x}(t), \quad h_1 < t \le h_2$$

 $z(t) = y(t), \quad h_2 \le t \le h_2 + \alpha.$

Claim: z is a solution of (1.5) on $h_1 < t \le h_2 + \alpha$. Since y is a unique solution of (1.5) on $h_2 - \alpha \le t \le h_2 + \alpha$, we have

$$\hat{x}(t) = y(t), \quad h_2 - \alpha \le t \le h_2$$

We note that z is a solution of (1.5) on $h_2 \leq t \leq h_2 + \alpha$ and so it only remains to verify that z' is continuous at the point $t = h_2$. Clearly,

$$z(t) = \hat{x}(h_2) + \int_{h_2}^{t} f(s, z(s))ds, \quad h_2 \le t \le h_2 + \alpha.$$
(1.20)

Further,

$$\hat{x}(h_2) = x_0 + \int_{t_0}^{h_2} f(s, z(s)) ds.$$
(1.21)
Thus, the relation (1.20) and (1.21) together yield **OPESA**
$$z(t) = x_0 + \int_{t_0}^{h_2} O(s, z(s)) ds + \int_{t_0}^{t} f(s, z(s)) ds$$
$$= x_0 + \int_{t_0}^{t} O(s, z(s)) ds, \qquad h_1 \le t \le h_2 + \alpha.$$

Obviously, the derivatives at the end points h_1 and $h_2 + \alpha$ are one-sided. We summarize :

Theorem 1.4.1. Let

- (i) $D \subset \mathbb{R}^{n+1}$ be an open connected set and let $f : D \to \mathbb{R}$ be continuous and satisfy the Lipschitz condition in x on D;
- (ii) f be bounded on D and
- (iii) x be a unique solution of the IVP (1.5) existing on $h_1 < t < h_2$.

Then,

$$\lim_{t \to h_2 = 0} x(t)$$

exists. If $(h_2, x(h_2 - 0)) \in D$, then x can be continued to the right of h_2 .

Since the function f is continuous on the rectangle

$$R = \left\{ (t, x) : |t - t_0| \le T, \ |x - x_0| \le \frac{L}{K} (e^{KT} - 1) \right\},\$$

there exists a real number $L_1 > 0$ such that

$$|f(t,x)| \le L_1, \ (t,x) \in R.$$

Moreover, the convergence of the sequence $\{x_n\}$ is uniform implies that the limit x is continuous. From the corollary (1.14), it follows that

$$|x(t) - x_n(t)| \le \frac{L_1}{K} \frac{(KT)^{n+1}}{(n+1)!} e^{KT}.$$

Finally, we show that x is a solution of the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s)) ds, \quad t_0 \le t \le t_0 + T.$$
(1.35)

Also

$$|x(t) - x_0 - \int_{t_0}^t f(s, x(s))ds| = |x(t) - x_n(t) + \int_{t_0}^t \left[f(s, x_n(s)) - f(s, x(s)) \right]ds|$$

$$\leq |x(t) - x_n(t)| + \int_{t_0}^t \left| f(s, x(t)) - f(s, x_n(s))ds \right|$$
(1.36)

Since $x_n \to x$ uniformly on $[t_0, t_0 + T]$, the right side of (1.36) tends to z for $s_n \to \infty$. By letting $n \to \infty$, from (1.36) we indeed have

se

$$|x(t) - x_0 - \int_{t_0}^t f(s,t(t))ds| \leq 0, \quad t \in [t, t + T].$$

 $x(t) = x_0 + \int_{t_0}^t f(s,t(s))ds, \quad t \in [t_0, t_0 + T].$

or else

The uniqueness of x follows similarly as shown in the proof of Theorem 1.3.2.

Remark : The example cited at the beginning of this section does not contradict the Theorem 1.5.1 as $f(t, x) = x^2$ does not satisfy the strip condition $f \in \text{Lip}(S, K)$.

A consequence of the Theorem 1.5.1 is :

Theorem 1.5.2. Assume that f(t, x) is a continuous function on $|t| < \infty$, $|x| < \infty$. Further, let f satisfies Lipschitz condition on the the strip S_a for all a > 0, where

$$S_a = \{(t, x) : |t| \le a, |x| < \infty\}.$$

Then, the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0,$$
 (1.37)

has a unique solution existing for all t.

Lecture 6

1.6 Existence and Uniqueness of Solutions of Systems

The methodology developed till now concerns existence and uniqueness of a single equation or usually called a scalar equations which is a natural extension for the study of a system of equations or to higher order equations. In the sequel, we glance at these extensions. Let $I \subseteq \mathbb{R}$ be an interval, $E \subseteq \mathbb{R}^n$. Let $f_1, f_2, ..., f_n : I \times E \to \mathbb{R}$ be given continuous functions. Consider a system of nonlinear equations



Denoting (column) vector x with components $x_1, x_2, ..., x_n$ and vector f with components $f_1, f_2, ..., f_n$, the system of equations (1.38) assumes the form

$$x' = f(t, x).$$
 (1.39)

A general *n*-th order equation is representable in the form (1.38) which means that the study of *n*-th order nonlinear equation is naturally embedded in the study O(1.69). It speaks of the importance of the study of systems of nonlinear constant, reacting O(1.69). It speaks of difficulties that one has to face. Consider an INC

and non-local existence theorems for systems of equations stated below (1.40)

The proceed of the and non-local existence theorems for systems of equations stated below have a remarkable resemblance to these of scalar equations. The detailed proofs are to be supplied by readers with suitable modifications to handle the presence of vectors and their norms. Below the symbol |.| is used to denote both the norms of a vector and the absolute value. There is no possibility of confusion since the context clarifies the situation.

In all of what follows we are concerned with the region D, a rectangle in \mathbb{R}^{n+1} space, defined by

$$D = \{(t, x) : |t - t_0| \le a, |x - x_0| \le b\},\$$

where $x, x_0 \in \mathbb{R}^n$ and $t, t_0 \in \mathbb{R}$.

Definition 1.6.1. A function $f: D \to \mathbb{R}^n$ is said to satisfy the Lipschitz condition in the variable x, with Lipschitz constant K on D if

$$|f(t,x_1) - f(t,x_2)| \le K|x_1 - x_2| \tag{1.41}$$

uniformly in t for all $(t, x_1), (t, x_2)$ in D.

Lecture 7

1.7 Cauchy-Peano Theorem

Let us recall that the IVP stated in Example 1.6.7 admits solutions. It is not difficult to verify, in this case, that f is continuous in (t, x) in the neighborhood of (0, 0). In fact, the continuity of f is sufficient to prove the existence of a solution. The proofs in this section are based on Ascoli-Arzela theorem which in turn needs the concept of equicontinuity of a family of functions. We need the following ground work before embarking on the proof of such results. Let $I = [a, b] \subset \mathbb{R}$ be an interval. Let $F(I, \mathbb{R})$ denote the set of all real valued functions defined on I.

Definition 1.7.1. A set $E \subset F(I, \mathbb{R})$ is called equicontinuous on I if for any $\epsilon > 0$, there is a $\delta > 0$ such that for all $f \in E$,

$$|f(x) - f(y)| < \epsilon$$
, whenever $|x - y| < \delta$.

Definition 1.7.2. A set $E \subset F(I, \mathbb{R})$ is called uniformly bounded on I if there is a M > 0, such that

$$|f(x)| < M$$
 for all $f \in E$ and for all $x \in I$.

Theorem 1.7.3. (Ascoli-Arzela Theorem) Let $B \subset F(I, \mathbb{R})$ be any uniformly bounded and equicontinuous set on I. Then, every sequence of functions $\{f_n\}$ in B contains a subsequence $\{f_{n_k}\}, k = 1, 2...,$ which converges uniformly on every compact sub-merce of I.

Theorem 1.7.4. (Peano's existence theorem) Let $\mathfrak{a} \in \mathbb{R}$. Let $S \subset \mathbb{R}^2$ be a strip defined by

Let
$$I : [t_0, t_0 + a]$$
 Let $Y \to \mathbb{R}$ be a bounded position function. Then, the IVP
 $x = f(t, x), \quad x(t_0) = x_0,$
(1.45)

has at least one solution existing on $[x_0 - a, x_0 + a]$.

Proof. The proof of the theorem is first dealt on $[t_0, t_0 + a]$ and the proof on $[t_0 - a, t_0]$ is similar with suitable modifications. Let the sequence of functions $\{x_n\}$ be defined by, for $n = 1, 2 \cdots$

$$x_n(t) = x_0, \quad t_0 \le t \le t_0 + \frac{a}{n}, \quad t \in I,$$

$$x_n(t) = x_0 + \int_{t_0}^{t - \frac{a}{n}} f(s, x_n(s)) ds \quad \text{if} \quad t_0 + \frac{ka}{n} \le t \le t_0 + \frac{(k+1)a}{n}, \quad k = 1, 2, \dots, n \quad (1.46)$$

We note that x_n is defined on $[t_0, t_0 + \frac{a}{n}]$ to start with and thereafter defined on

$$\left[t_0 + \frac{ka}{n}, t_0 + \frac{(k+1)a}{n}\right], \quad k = 1, 2, \dots, n.$$

- 1. Find the general solution of x''' + x'' + x' + x = 1 given that $\cos t$, $\sin t$ and e^{-t} are three linearly independent solutions of the corresponding homogeneous equation. Also find the solution when x(0) = 0, x'(0) = 1, x''(0) = 0.
- 2. Use the method of variation of parameter to find the general solution of x''' x' = d(t) where

(i) d(t) = t, (ii) $d(t) = e^t$, (iii) $d(t) = \cos t$, and (iv) $d(t) = e^{-t}$. In all the above four problems assume that the general solution of x''' - x' = 0 is $c_1 + c_2 e^{-t} + c_3 e^t$.

3. Assuming that $\cos Rt$ and $\frac{\sin Rt}{R}$ form a linearly independent set of solutions of the homogeneous part of the differential equation $x'' + R^2x = f(t), R \neq 0, t \in [0, \infty)$, where f(t) is continuous for $0 \leq t < \infty$ show that a solution of the equation under consideration is of the form

$$x(t) = A\cos Rt + \frac{B}{R}\sin Rt + \frac{1}{R}\int_0^t \sin[R(t-s)]f(s)ds,$$

where A and B are some constants. Show that particular solution of (2.14) is not unique. (Hint : If x_p is a particular solution of (2.14) and x is any solution of (2.15) then show that $x_p + cx$ is also a particular solution of (2.14) for any arbitrary constant c.)

Two Useful Formulae

Two formulae proved below are interesting in themselves. They are also useful while studying boundary value problems of second order equations. Consider an equation

$$L(y) = a_0(t)y'' + a_1(t)y' + a_2(t) \in \mathcal{S} \subset I,$$

where $a_0, a_1, a_2 : I \to \mathbb{R}$ are continuous familions in addition $a_0 \to \neq 0$ for $t \in I$. Let u and v be any two twice differentiable uncount on I. Consider

$$\mathbf{Pre}^{\mathbf{v}} uL(v) - vL(u) \mathbf{Q} \mathbf{Q} uv' \mathbf{v} u'') + a_1(uv' - vu'). \tag{2.23}$$

The Wronskian of u and v is given by W(u, v) = uv' - vu' which shows that

$$\frac{d}{dt}W(u,v) = uv'' - vu''.$$

Note that the coefficients of a_0 and a_1 in the relation (2.23) are W'(u, v) and W(u, v) respectively. Now we have

Theorem 2.4.5. If u and v are twice differential functions on I, then

$$uL(v) - vL(u) = a_0(t)\frac{d}{dt}W[u, v] + a_1(t)W[u, v], \qquad (2.24)$$

where L(x) is given by (2.7). In particular, if L(u) = L(v) = 0 then W satisfies

$$a_0 \frac{dW}{dt}[u, v] + a_1 W[u, v] = 0.$$
(2.25)

Lecture 13

The results which have been discussed above for a second order have an immediate generalization to a *n*-th order equation (2.34). The characteristic equation of (2.34) is given by

$$L(p) = a_0 p^n + a_1 p^{n-1} + \dots + a_n = 0.$$
(2.39)

If p_1 is a real root of (2.39) then, $e^{p_1 t}$ is a solution of (2.34). If p_1 happens to be a complex root, the complex conjugate of p_1 *i.e.*, \bar{p}_1 is also a root of (2.39). In this case

 $e^{at}\cos bt$ and $e^{at}\sin bt$

are two linearly independent solutions of (2.34), where a and b are the real and imaginary parts of p_1 , respectively.

We now consider when roots of (2.39) have multiplicity(real or complex). There are two cases:

(i) when a real root has a multiplicity m_1 ,

Then, as i

(ii) when a complex root has a multiplicity m_1 .

Case 1: Let q be the real root of (2.39) with the multiplicity m_1 . By induction we have m_1 linearly independent solutions of (2.34), namely

$$e^{qt}, te^{qt}, t^2 e^{qt}, \cdots, t^{m_1-1} e^{qt}.$$

Case 2: Let *s* be a complex root of (2.39) with the multiplicative r_1 : Let $s = s_1 + is_2.$
en, as in Case 1, we note that
 $e^{st} t^{(S_1, \ldots, t^{m_1-1}e^{st})}, t^{m_1-1} e^{st}, t^{(S_1, \ldots, t^{m_1-1}e^{st})}$
(2.40)

complex valued solutions of (2.34). For (2.34), the real and are m_1 linearly in \mathbf{c} imagin \mathbf{p} a collect solution \mathbf{p} (1) (2.40) is also a solutions of (2.34). So in this case $2m_1$ linearly independent solutions of (2.34) are given by

$$\left. \begin{array}{c} e^{s_{1}t}\cos s_{2}t, \quad e^{s_{1}t}\sin s_{2}t \\ te^{s_{1}t}\cos s_{2}t, \quad te^{s_{1}t}\sin s_{2}t \\ t^{2}e^{s_{1}t}\cos s_{2}t, \quad t^{2}e^{s_{1}t}\sin s_{2}t \\ \dots \\ t^{m_{1}-1}e^{s_{1}t}\cos s_{2}t, \quad t^{m_{1}-1}e^{s_{1}t}\sin s_{2}t \end{array} \right\}$$

$$(2.41)$$

Thus, if all the roots of the characteristic equation (2.39) are known, no matter whether they are simple or multiple roots, there are n linearly independent solutions and the general solution of (2.34) is

$$c_1x_1 + c_2x_2 + \dots + c_nx_n$$

where x_1, x_2, \dots, x_n are n linearly independent solutions and c_1, c_2, \dots, c_n are any constants. To summarize :

Lecture 15

Example 3.2.4. For illustration we consider a linear equation

$$x''' - 6x'' + 11x' - 6x = 0$$

Denote

$$x_1 = x$$
, $x'_1 = x_2 = x'$, $x'_2 = x'' = x_3$.

Then, the given equation is equivalent to the system x' = A(t)x, where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}.$$

The required general solution of the given equation x_1 and in the present case, we see that

$$x_1(t) = c_1 e^t + c_2 e^{2t} + c_3 e^{3t},$$

where c_1, c_2 and c_3 are arbitrary constants.

EXERCISES

- 1. Find a system of first order differential equation for which y defined by
- $\sum_{v_2^2+3, v_2'=0}^{-1} \sum_{v_2}^{-1} \sum_{v_2'=0}^{-1} \sum_{v_2$ is a solution. 2. Represent the IVP \mathbf{as} $x' = f(t, x), \ x(0) = x_0.$

Show that f is continuous. Find a value of M such that

$$|f(t,x)| \le M$$
 on $R = \{(t,x) : |t| \le 1, |x| \le 1\}.$

3. The system of three equations is given by

$$(x_1, x_2, x_3)' = (4x_1 - x_2, 3x_1 + x_2 - x_3, x_1 + x_3).$$

Then,

- (i) show that the above system is linear in x_1, x_2 and x_3 ;
- (ii) find the solution of the system.
- 4. On the rectangle R

Substituting the values of $\phi'_{11}, \phi'_{12}, \cdots, \phi'_{1n}$ from (3.19), the first term on the right side reduces to

$$\begin{vmatrix} \sum_{k=1}^{n} a_{1k}\phi_{k1} & \sum_{k=1}^{n} a_{1k}\phi_{k2} & \cdots & \sum_{k=1}^{n} a_{1k}\phi_{kn} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}$$

which is $a_{11} \det \Phi$. Carrying this out for the remaining terms it is seen that

 $(\det \Phi)' = (a_{11} + a_{22} + \dots + a_{nn})\det\Phi = (trA)\det\Phi.$

The equation thus obtained is a linear differential equation. The proof of the theorem is complete since we know that the required solution of this is given by (3.18).

Theorem 3.3.2. A solution matrix Φ of (3.16) on I is a fundamental matrix of (3.14) on I if and only if det $\Phi \neq 0$ for $t \in I$.

Proof. Let Φ be a solution matrix such that det $\Phi(t) \neq 0, t \in I$. Then, the columns of Φ are linearly independent on I. Hence, Φ is a fundamental matrix. The proof of he converse is still easier and hence omitted.

Some useful properties of the fundamental matrix are established below.

Theorem 3.3.3. Let Φ be a fundamental matrix for the system (3.14) and let C be a constant non-singular matrix. Then, ΦC is also a fundamental matrix for (3.14). In Alition, every fundamental matrix Ψ of (3.14) is ΦC for some non-singular metric C.

Proof. The first part of the theorem is a single of sequence of Theorem 3.3.2 and the fact that the product of non-singular matrices is non-singular. Let Φ_1 and Φ_2 be two fundamental matrices for (3.14) and let $\Phi_2 = \Phi_1 \Psi$. Then $\Phi'_2 = \Phi_1 \Psi \oplus \Phi'_1 \Psi$. Equation (3.16) now implies that $A\Phi_2 = \Phi_1 \Psi' + i\Phi_2 \Psi \oplus \Phi_1 \Psi' + A\Phi_2$. Then $\Phi_1 \Psi' = 0$ which shows that $\Psi' = 0$. Hence, $\Psi = 0$, where C is a constant fraction. Since Φ_1 and Φ_2 are non-singular so is C. \Box

EXERCISES

- 1. Let Φ be a fundamental matrix for (3.14) and C is any constant non-singular matrix then, in general, show that $C\Phi$ need not be a fundamental matrix.
- 2. Let $\Phi(t)$ be a fundamental matrix for the system (3.14), where A(t) is a real matrix. Then, show that the matrix $(\Phi^{-1}(t))^T$ satisfies the equation

$$\frac{d}{dt}(\Phi^{-1})^T = -A^T(\Phi^{-1})^T,$$

and hence show that $(\Phi^{-1})^T$ is a fundamental matrix for the system

$$x' = -A^T(t)x, \ t \in I.$$
 (3.23)

System (3.23) is called the "adjoint" system to (3.14) and vice versa.

- 3. Let Φ be a fundamental matrix for Eq.(3.14), with A(t) being a real matrix. Then, show that Ψ is a fundamental matrix for its adjoint (3.23) if and only if $\Psi^T \Phi = C$, where C is a constant non-singular matrix.
- 4. Consider a matrix P defined by

$$P(t) = \begin{bmatrix} f_1(t) & f_2(t) \\ 0 & 0 \end{bmatrix}, \ t \in I,$$

where f_1 and f_2 are any two linearly independent invariants on I. Then, show that $det[P(t)] \equiv 0$ on I, but the columns of P(t) is a mean properties. Can the columns P be solutions of linear homogeneous systems of equations of the form (3.14)? (See it in the light of Theorem 4.13)

- 5. Each the element of fundamentar matrix Φ which satisfies $\Phi(0) = E$ for the system (a) $A = \begin{bmatrix} -1 & 3 & 4 \\ 0 & 2 & 0 \\ 1 & 5 & -1 \end{bmatrix};$ (b) $A = \begin{bmatrix} 1 & 3 & 8 \\ -2 & 2 & 1 \\ -3 & 0 & 5 \end{bmatrix}$
- 6. Can the following matrices Φ be candidates for fundamental matrices for some linear system

$$x' = A(t)x, \ t \in I,$$

3.6 Phase Portraits-Introduction Lecture 19

In this elementary study, we wish to draw the phase portraits for a system of two linear ordinary differential equations. In order to make life easy, we first go through a bit of elementary linear algebra.Parts A and B are more or less a revision ,,which hopefully helps the readers to draw the phase portraits. We may skip Parts A and B in case we are familiar with curves and elementary canonical forms for real matrices.

Part A: Preliminaries.

Let us recall: \mathbb{R} denotes the real line. By \mathbb{R}^n , we mean the standard or the usual Euclidean space of dimension $n, n \geq 1$. A $n \times n$ matrix A is denoted by $(a_{ij})_{n \times n}, a_{ij} \in \mathbb{R}$. The set of all such real matrices is denoted by $M_n(\mathbb{R})$. $A \in M_n(\mathbb{R})$ also induces a linear operator on \mathbb{R}^n (now understood as column vectors) defined by

$$\stackrel{A}{\rightarrow} A(x) \text{ or } A \colon \mathbb{R}^n \to \mathbb{R}^n$$

more explicitly defined by A(x) = Ax(matrix multiplication). $L(\mathbb{R}^n)$ denotes the set of all linear transformations from \mathbb{R}^n to \mathbb{R}^n . For a $n \times n$ real matrix A, we some times use $A \in M_n(\mathbb{R})$ or $A \in L(\mathbb{R}^n)$. Let $T \in L(\mathbb{R}^n)$. Then, Ker(T) or N(T) (read as kernel of T or Null space of T ejectively) is defined by

$$Ker(T) = N(T): = \{x \in \mathbb{R}^n : Tx = o\}$$

The dimension of Ker(T) is called the nullity of T and C and C and $\nu(T)$. The dimension of range of T is called the rank of T and is detected by $\rho(T)$. For any $T \in L(\mathbb{R}^n)$ the Rank Nullity Theorem asserts

Consequently $\mathbb{P}: \mathbf{V} \in L(\mathbb{R}^n)$ (i.e. $T \in \mathbb{R}^n \in \mathbb{R}^n$ is linear.) T is one-one iff T is onto. Let us now prove the following result T is a specific transmission of the following result T is a specific transmission of the following result T is a specific transmission of the following result T is a specific transmission of the following result T is a specific transmission of the following result T is a specific transmission of the following result T is a specific transmission of the following result T is a specific transmission of transm

1. **Theorem** : Given $T \in L(\mathbb{R}^n)$ and given $t_0 \ge 0$, the series

x

$$\sum_{k=0}^{\infty} \frac{T^k}{k!} t^k$$

ı

is absolutely and uniformly convergent for all $|t| \le t_0$. **Proof**: We let ||T|| = a. We know

$$\| \frac{T^k t^k}{k!} \| \le \frac{a^k t_0^k}{k!}$$

and

$$\sum_{k=0}^{\infty} \frac{a^k t_0^k}{k!} = e^{a t_0}$$



Figure 3.2:

while the phase portrait of (3.49) is (fig.3.3)



Figure 3.3:

Supply the details for drawing Figures 3.2 and 3.3.

In general, it is easy to write/draw the phase portrait of (3.49) when A in its canonical form. Coming back to (3.49), let P be an invertible 2×2 matrix such that $B = P^{-1}AP$, where B is a canonical form of A. We now consider the system

$$y' = By \tag{3.51}$$

By this time it is clear that phase portrait for (3.49) is the phase portrait of (3.51) under the transformation x = Py. We also write that B has one of the following form.

(a)
$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$
 (b) $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ (c) $B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$



Figure 3.7:

Also, we note that the phase portraits for (??) is a family of ellipses as shown in Figure 8.



Figure 3.8:

In this case the origin is called the center for the system (??). We end this short discussion with an example.

Example : Consider the linear system

$$\dot{x}_1 = -4x_2 \; ; \; \dot{x}_2 = x_1$$

$$or \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} ; \; A = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$$

It is easy to verify that A has two non-zero (complex) eigenvalues $\pm 2i$. With usual notations

- 3. Prove that any solution x of (4.2) has at most a countable number of zeros in $(0, \infty)$.
- 4. Show that the equation

$$x'' + a(t)x' + b(t)x = 0, \quad t \ge 0$$
 (*)

transforms into an equation of the form

$$(p(t)x')' + q(t)x = 0, \quad t \ge 0$$
 (**)

by multiplying (*) throughout by $\exp(\int_0^t a(s)ds)$, where a and b are continuous functions on $[0, \infty)$,

$$p(t) = \exp(\int_0^t a(s)ds)$$
 and $q(t) = b(t)p(t)$.

State and prove a result similar to Theorem 4.1.5 for equation (*) and (**). Also show that if $a(t) \equiv 0$, then, (**) reduces to $x'' + q(t)x = 0, t \ge 0$.



- (a) Either $A = -\infty$ or $B = \infty$.
- (b) Both $A = -\infty$ and $B = \infty$.
- (c) a(t) = 0 for at least one point t in (A, B).

The proof is obvious.

In this chapter, the discussions are confined to only regular BVPs. The definitions listed so far lead to the definition of a nonlinear BVP.

Definition 4.4.8. A BVP which is not a linear BVP is called a nonlinear BVP.

A careful analysis of the above definition shows that the nonlinearity in a BVP may be introduced because

- (i) the differential equation may be nonlinear;
- (ii) the given differential equation may be linear but the boundary conditions may not be linear homogeneous.

The assertion made in (i) and (ii) above is further clarified in the following example.

Example 4.4.9. (i) The BVP

$$x'' + |x| = 0, \quad 0 < t < \pi$$
with boundary conditions $x(0) = x(\pi) = 0$ is not linear due to the presence of $|x|$.
(ii) The BVP
with boundary conditions
 $x(0).x(1) = x(0); \quad x'(0) = 0$
is a bunk real BVP since one of the boundary conditions is not linear homogeneous.
EXERCISES

- 1. State with reasons whether the following BVPs are linear homogeneous, linear non-homogeneous or non-linear.
 - (i) $x'' + \sin x = 0$, $x(0) = x(2\pi) = 0$. (ii) x'' + x = 0, $x(0) = x(\pi)$, $x'(0) = x'(\pi)$. (iii) $x'' + x = \sin 2t$, $x(0) = x(\pi) = 0$. (iv) $x'' + x = \cos 2t$, $x^2(0) = 0$, $x^2(\pi) = x'(0)$.
- 2. Are the following BVPs regular ?
 - (i) 2tx'' + x' + x = 0, x(-1) = 1, x(1) = 1.
 - (ii) 2x'' 3x' + 4x = 0, $x(-\infty) = 0$, x(0) = 1.
 - (iii) x'' 9x = 0, x(0) = 1, $x(\infty) = 0$.

Lecture 27

The following theorem deals with periodic boundary conditions given in (4.20).

Theorem 4.5.5. Let the assumptions of theorem ?? be true. Suppose x_m and x_n are eigenfunctions of BVP (4.17) and (4.20) corresponding to the distinct eigenvalues λ_m and λ_n respectively. Then, x_m and x_n are orthogonal with respect to the weight function r(t).

Proof. In this case

$$\left[pW(x_n, x_m)\right]_A^B = p(B)[x_n(B)x'_m(B) - x'_n(B)x_m(B) - x_n(A)x'_m(A) + x'_n(A)x_m(A)].$$

The expression inside the brackets is zero once we use the periodic boundary condition (4.20) .

The following theorem ensures that the eigenvalues of (4.17), (4.18) or (4.17), (4.19) are real if r > 0 (or r(t) < 0) on (A, B) and r is continuous on [A, B].

Theorem 4.5.6. Let the hypotheses of Theorem ?? hold. Suppose that r is positive on (A, B) or r is negative on (A, B) and r is continuous on [a, B]. Then, all the eigenvalues of BVP(4.17), (4.18) or (4.17), (4.19) are real.

Proof. Let $\lambda = a + ib$ be an eigenvalue and let

$$x(t) = m(t) + in(t)$$

be a corresponding eigenfunction, where a, b, m(t) and n(t) are real. From (4.17) we have

$$(pm' + pin')' + q(m + in) + (a + ib)r(m + in) = 0$$

Equating the real and imaginary parts, we have

and

$$(pn')' + (q + 4r)w + 2rm = 0.$$

Elimination of $(q + ar)$ in the above two equations implies
 $-b(m^2 + n^2)r = m(pn')' - n(pm')' = \frac{d}{dt}[(pn')m - (pm')n]$

(pm')' + (q + a)

Thus, by integrating, we get

$$-b\int_{A}^{B} (m^{2}(s) + n^{2}(s))r(s)ds = \left[(pn')m - (pm')n\right]_{A}^{B}.$$
(4.26)

Since m and n satisfy one of the boundary conditions (4.18) and (4.19) or (4.20), we have, as shown earlier,

$$\left[p(n'm - m'n)\right]_{A}^{B} = \left[pW(m,n)\right]_{A}^{B} = 0.$$
(4.27)

Also

$$\int_{A}^{B} [m^2(s) + n^2(s)]r(s)ds \neq 0$$

by the assumptions. Hence, from (4.26) and (4.27) it follows that b = 0, which means that λ is real which completes the proof.

$$g(t) = c_0 p_0(t) + c_1 p_1(t) + \dots + c_n p_n(t) + \dots,$$

where

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} g(s) P_n(s) ds, n = 0, 1, 2, \cdots$$

since

$$\int_{-1}^{1} P_n^2(s) ds = \frac{2}{2n+1}, n = 0, 1, 2, \cdots$$
EXERCISES

- 1. Show that corresponding to an eigenvalue the Sturm-Liouville problem (4.17), (4.18) or (4.17), (4.19) has a unique eigenfunction.
- 2. Show that the eigenvalues for the BVP

$$x'' + \lambda x = 0, x(0) = 0$$
 and $x(\pi) + x'(\pi) = 0$

satisfy the equation

$$\sqrt{\lambda} = -\tan \pi \sqrt{\lambda}.$$

Prove that the corresponding eigenfunctions are

where
$$\lambda_n$$
 is an eigenvalue.
3. Consider the equation
Find the eigenvalue and eigenfunctions for the following cases:
 $(1, C_0) = x'(\pi) = 0;$
 $(i) x(0) = 0, x'(\pi) = 0;$
 $(ii) x(0) = x(\pi) = 0;$
 $(iv) x'(0) = x(\pi) = 0.$

With this choice of c_1 and c_2 , G(t, s) defined by the relation (4.35) has all the properties of the Green's function. Since y and z satisfy L(x) = 0 it follows that

$$y(pz')' - z(py')' \equiv \frac{d}{dt}[p(yz' - y'z)] = 0.$$
(4.37)

Hence

$$p(t)[y(t)z'(t) - y'(t)z(t)] = A \text{ for all } t \text{ in } [a, b]$$

where A is a non-zero constant (because y and z are linearly independent solutions of L(x) = 0). In particular it is seen that

$$y(s)z'(s) - y'(s)z(s)] = A/p(s), A \neq 0$$
(4.38)

From equation (4.36) and (4.38) it is seen that

$$c_1 = -z(s)/A, c_2 = -y(s)/A.$$

Hence the Green's function is

$$G(t,s) = \begin{cases} -y(t)z(s)/A & \text{if } t \le s, \\ -y(s)z(t)/A & \text{if } t \ge s. \end{cases}$$

$$(4.39)$$

. . **k**

The main result of this article is Theorem .

Theorem 4.6.2. Let
$$G(t, s)$$
 be given by the relation (4.39) then $x(t)$ is a colution of (4.32),
(4.33) and (4.34) if and only if
 $f(0+)_a G(t,s)f(s)ds$ (4.40)
Proof. Let the electron (4.40) hold. The
 $x(t) = -\left[\int_a^b z(t)y(s)f(s)ds + \int_t^b y(t)z(s)f(s)ds\right]/A.$ (4.41)

Differentiating (4.41) with respect to t yields

$$x'(t) = -\left[\int_{a}^{t} z'(t)y(s)f(s)ds + \int_{t}^{b} y'(t)z(s)f(s)ds\right] / A.$$
(4.42)

Next on computing (px')' from (4.42) and adding to qx in view of y and z being solutions of L(x) = 0 it follows that

$$L(x(t)) = -f(t)$$
 (4.43)

Further, from the relations (4.41) and (4.42), it is seen that

$$\begin{cases} Ax(a) = -y(a) \int_{a}^{b} z(s)f(s)ds, \\ Ax'(a) = -y'(a) \int_{a}^{b} z(s)f(s)ds. \end{cases}$$
(4.44)

and $\eta \in \mathbb{R}$ be a number such that

$$\alpha_j < \eta, \quad (j = 1, 2, \cdots, m). \tag{5.3}$$

Then, there exists a real constant M > 0 such that

$$|e^{At}| \le M e^{\eta t}, \quad 0 \le t < \infty.$$
(5.4)

Proof. Let e_j be the *n*-vector with 1 in the *j*-th place and zero elsewhere. Then,

$$\varphi_j(t) = e^{At} e_j, \tag{5.5}$$

 \cdots, n).

denotes the *j*-th column of the matrix e^{At} . From the previous module on systems of equations, we know that

$$e^{At}e_j = \sum_{r=1}^m (c_{r1} + c_{r2}t + \dots + c_{rn_r}t^{n_r-1})e^{\lambda_r t},$$
(5.6)

where $c_{r1}, c_{r2}, \dots, c_{rn_r}$ are constant vectors. From (5.5) and (5.6) we have

$$|\varphi_j(t)| \le \sum_{r=1}^m (|c_{r1}| + |c_{r2}|t + \dots + |c_{rn_r}|t^{n_r-1})|\exp(\alpha_r + i\beta_r)t| = \sum_{r=1}^m P_r(t)e^{\alpha_r t}$$
(5.7)

where P_r is a polynomial in t. By (5.3),

(5.8) so
$$M_j > 0$$
 such that

for sufficiently large values of t. In view of (5.7) and (5.8), here exists $M_j > 0$

pre

$$|e^{At}| \le \sum_{j=1}^{n} |\varphi_j(t)| \le (M_1 + M_2 + \dots + M_n)e^{\eta t} = Me^{\eta t} \quad (0 \le t < \infty),$$

where $M = M_1 + M_2 + \cdots + M_n$ which proves the inequality (5.4).

Actually we have estimated an upper bound for the fundamental matrix e^{At} for the equation (5.1) in terms of an exponential function through the inequality (5.4). Theorem 5.2.2 proved subsequently is a direct consequence of Theorem 5.3.1. It tells us about a necessary and sufficient conditions for the solutions of (5.1) decaying to zero as $t \to \infty$. In other words, it characterizes a certain asymptotic behavior of solutions of (5.1) It is quite easy to sketch the proof and so the details are omitted.

Theorem 5.2.2. Every solution of the equation (5.1) tends to zero as $t \to +\infty$ if and only if the real parts of all the eigenvalues of A are negative.

Obviously, if the real part of an eigenvalue is positive and if φ is a solution corresponding to this eigenvalue then,

$$|\varphi(t)| \to +\infty$$
, as $t \to \infty$.

We shift our attention to the system

$$x' = Ax + b(t), \tag{5.9}$$

where A is an $n \times n$ constant matrix, is a perturbed system with a perturbation term b, where $b : [0, \infty) \to \mathbb{R}$ is assumed to be continuous. Since a fundamental matrix for the system (5.1) is e^{tA} any solution of (5.9) is (by the method of variation of parameters) is

$$x(t) = e^{(t-t_0)A}x_0 + \int_{t_0}^t e^{(t-s)A}b(s)ds, \quad t \ge t_0 \ge 0,$$

satisfies the equation (5.9). Here x_0 is an *n*- (column)vector such that $x(t_0) = x_0$. we take the norm on both sides to get

$$|x(t)| \le |e^{(t-t_0)A}x_0| + \int_{t_0}^t |e^{(t-s)A}| |b(s)| ds, \quad 0 \le t_0 \le t < \infty.$$

Suppose $|x_0| \leq K$ and η is a number such that

$$\eta > R \exp(\text{real part of } \lambda_i), \ i = 1, 2, \cdots, m,$$

where λ_i are the eigenvalues of the matrix A. Now, in view of (5.1) we have

$$|x(t)| \le KMe^{\eta(t-t_0)} + 11 \sqrt{\bigcup_{t_0}^{t} h^{t_{sort}}} |b(s)| ds$$
(5.10)

The inequality (5.10) is a consequence of Theorem 5. (1) We note that the right side is independent of x. It does not be constant K, W and η and the function b. The inequality (5.10) **5** pointwise estimate. The network of x for large values of t depends on the sign of the constant φ and the function t. In the following result, we assume that b satisfies a certain growth condition which yield an estimate for x.

Theorem 5.2.3. Suppose that b satisfies

$$|b(t)| \le p e^{at}, \quad t \ge T \ge 0, \tag{5.11}$$

where p and a are constants with $p \ge 0$. Then, every solution x of the system (5.9) satisfies

$$|x(t)| \le Le^{qt} \tag{5.12}$$

where L and q are constants.

Proof. Since b is continuous on $0 \le t < \infty$, every solution x(of (5.9)) exists on $0 \le t < \infty$. Further

$$x(t) = e^{At}c + \int_0^t e^{(t-s)A}b(s)ds, \quad 0 \le t < \infty,$$
(5.13)

Hence, Φ is uniformly bounded on $[0, \infty)$. The condition in (5.26) implies, as in Theorem 5.3.5, that $\Phi^{-1}(t)$ is bounded. Taking the norm on either side we have

$$|\varphi(t)| \le |\Phi(t)||C| + |\Phi(t)| \int_0^t |\Phi^{-1}(s)||b(s)|ds|$$

Now each term on the right side is bounded which shows that $\varphi(t)$ is also bounded.

EXERCISES

1. Show that any solution of x' = A(t)x tends to zero as $t \to 0$ where,

(i)
$$A(t) = \begin{bmatrix} -t & 0 & 0 \\ 0 & -t^2 & 0 \\ 0 & 0 & -t^2 \end{bmatrix};$$

(ii) $A(t) = \begin{bmatrix} -e^t & -1 & -\cos t \\ 1 & -e^{2t} & t^2 \\ \cos t & -t^2 & -e^{3t} \end{bmatrix};$
(iii) $A(t) = \begin{bmatrix} -t & \sin t \\ 0 & e^{-t} \end{bmatrix}.$

2. Let x be any solution of a system x' = A(t)x. Let M(t) be the largest eigenvalue of

$$A(t) + A^{T}(t)$$
 such that $\int_{t_0}^{\infty} M(s)ds < \infty$.

Show that x is bounded.

- 3. Prove that all the solutions of x' = A(t) are bounded, where A(t) is given by (i) $\begin{bmatrix} e^t & -1 & -2 \\ 1 & e^{-2t} & 2 \end{bmatrix}$, (ii) $\begin{bmatrix} (1+t)^{-2} & \sin t & 0 \\ -\sin t & 2 & 0 \end{bmatrix}$ and (iii) $\begin{bmatrix} e^{-t} & 0 \\ 0 & -1 \end{bmatrix}$.
- $\mathbf{P} \mathbf{r} \mathbf{e}^{-3t} \mathbf{P} \mathbf{a} \mathbf{G} \mathbf{e}^{-3t} \mathbf{e}^{$
- 4. What can you say about the boundedness of solutions of the system

$$x' = A(t)x + f(t) \text{ on } (0, \infty)$$

when a particular solution x_p , the matrix A(t) and the function f are given below:

(i)
$$x_p(t) = \begin{bmatrix} e^{-t} \sin t \\ e^{-t} \cos t \end{bmatrix}$$
, $A(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $f(t) = \begin{bmatrix} e^{-t} & \cos t \\ -e^{-t} & \sin t \end{bmatrix}$,
(ii) $x_p(t) = \begin{bmatrix} \frac{1}{2}(\sin t - \cos t) \\ 0 \\ 0 \end{bmatrix}$, $A(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -t^2 & 0 \\ 0 & 0 & -t^2 \end{bmatrix}$, $f(t) = \begin{bmatrix} \sin t \\ 0 \\ 0 \end{bmatrix}$.

5. Show that the solutions of

$$x' = A(t)x + f(t)$$

are bounded on $[0,\infty)$ for the following cases:

- 3. Is the origin stabile in the following cases:
 - (i) x''' + 6x'' + 11x' + 6x = 0, (ii) x''' - 6x'' + 11x' - 6x = 0,
 - (iii) x''' + ax'' + bx' + cx = 0, for all possible values of a, b and c.
- 4. Consider the system

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}' = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Show that no non-trivial solution of this system tends to zero as $t \to \infty$. Is every solution bounded? Is every solution periodic?

- 5. Prove that for $1 < \alpha < \sqrt{2}$, $x' = (\sin \log t + \cos \log t \alpha)x$ is asymptotically stable.
- 6. Consider the equation

$$x' = a(t)x.$$

Show that the origin is asymptotically stable if and only if

$$\int_0^\infty a(s)ds = -\infty.$$

Under what condition the zero solution is stable ?

sale.co.uk Stability of Linear and Qu 5.5

In this section the stability of line and a class more focus on linear sector h. Needless to stress to and a class of quasil near systems are discussed with importance of these topics as these have wide appicut c.s. Many physical radius have a represention through (5.32), which may be written in a more useful Crrc.

$$x' = A(t)x + f(t, x).$$
(5.35)

The equation (5.35) simplifies the work since it is closely related with the system

$$x' = A(t)x. \tag{5.36}$$

The equation (5.35) is perturbed form of (5.35). Many properties of (5.36) have already been discussed. Under some restrictions on A and f, stability properties of (5.35) are very similar to those of (5.36). We assume, to proceed further,

- (i) Let us recall : $I = [t_0, \infty)$, for $\rho > 0$, $S_{\rho} = \{x \in \mathbb{R}^n : |x| < \rho\}$.
- (ii) the matrix A(t) is an $n \times n$ matrix which is continuous on I;
- (iii) $f: I \times S_{\alpha} \to \mathbb{R}^n$ is a continuous function with $f(t, 0) \equiv 0, t \in I$.

which takes the form

$$|y(t)|e^{\rho t} \le M|y_0|e^{\rho t_0} + M \int_{t_0}^t e^{\rho s} |f(s, y(s))| ds.$$

Let $|y_0| < \alpha$. Then, the relation (5.42) is true in any interval $[t_0, t_1)$ for which $|y(t)| < \alpha$. In view of the condition (5.41), for a given $\epsilon > 0$ we can find a positive number δ such that

$$|f(t,x)| \le \epsilon |x|, \quad t \in I, for|x| < \delta.$$
(5.44)

Let us assume that $|y_0| < \delta$. Then, there exists a number T such that $|y(t)| < \delta$ for $t \in [t_0, T]$. Using (5.44) in (5.43), we obtain

$$e^{\rho t}|y(t)| \le M|y_0|e^{\rho t_0} + M\epsilon \int_{t_0}^t e^{\rho s}|y(s)|ds,$$
 (5.45)

for $t_0 \leq t < T$. An application of Gronwall's inequality to (5.45), yields

$$e^{\rho t}|y(t)| \le M|y_0|e^{\rho t_0}.e^{M\epsilon(t-t_0)}$$
(5.46)

or for $t_0 \leq t < T$, we obtain

Choose $M\epsilon < \rho$ and $y(t_0) =$

$$|y(t)| \le M |y_0| e^{(M \epsilon - \rho)(t - t_0)}.$$

 $y_0. \text{ If } |y_0| < \delta/M, \text{ then, (5.47) yields COUV47}$
 $|y(t)| < \delta, \text{ the OES}$

The solution y of the equation (C5) exists locally areach point $(t, y), t \ge t_0, |y| < \alpha$. Since the function f is defined on $I \times S_\alpha$, we extend the solution y interval by interval by preserving its probability δ . So given an collution $y(t) = y(t; t_0, y_0)$ with $|y_0| < \delta/M$, y exists on $t_0 \le t < \infty$ and satisfies $\psi_0(t) \ll \delta$. In the above discussion, δ can be made arbitrarily small. Hence, $y \equiv 0$ is asymptotically stable when $M\epsilon < \rho$.

When the matrix A is a function of t (ie A is not a constant matrix), still the stability properties solutions of (5.35) and (5.36) are shared but now the fundamental matrix needs to satisfy some stronger conditions. Let $r: I \to \mathbb{R}^+$ be a non-negative continuous function such that

$$\int_{t_0}^{\infty} r(s) ds < +\infty$$

Let f be continuous and satisfy the inequality

$$|f(t,x)| \le r(t)|x|, (t,x) \in I \times S_{\alpha}, \tag{5.48}$$

The condition (5.48) guarantees the existence of a null solution of (5.35). Now the following is a result on asymptotic stability of the zero solution of (5.35).

A consequence :

In the light of the above proposition we again repeat that that the positive definiteness of the matrices R and Q is a necessary and sufficient condition for the asymptotic stability of the zero solution of the linear system (5.60).

Remark: The stability properties of zero solution of the equation (5.62) is unaffected if the system (5.60) is transformed by the relation x = Py, where P is a non-singular constant matrix. The system (5.60) then transforms to

$$y' = (P^{-1}AP)y.$$

Now choose the matrix P such that

$$P^{-1}AP$$

is a triangular matrix. Such a transformation is always possible by Jordan normal form. So there is no loss of generality by assuming in (5.60) that, the matrix A is such that its main diagonal consists of eigenvalues of A and for i < j, $a_{ij} = 0$. In other words the matrix A is of the following form:



Equating the corresponding terms on both sides results in the following system of equations

$$(\lambda_j + \lambda_k)r_{jk} = -q_{jk} + \delta_{jk}(\cdots, r_{hk}, \cdots),$$

where δ_{jk} is a linear form in r_{hk} , h+k > j+k, with coefficients in a_{rs} . Hopefully the above system determines r_{jk} . The solution of the linear system is unique if the determinant of the