§15.1 Vector Fields

1. Define a vector field: Assigns a vector to each point in the plane or in 3-space. Can be visualized as loads of arrows. Can represent a force field or fluid flow - both are useful.

2. Two important definitions. Often before I do these I define $\nabla = \frac{\partial}{\partial x}\,\hat{\imath} + \frac{\partial}{\partial y}\,\hat{\jmath} + \frac{\partial}{\partial z}\,\hat{k}$ so that gradient, divergence and curl all make sense with how $\nabla$ is used.

   (a) The *divergence* $\nabla \cdot \bar{F} = M_x + N_y + P_z$ gives the net fluid flow in/out of a point (very small ball).

   (b) The *curl* $\nabla \times \bar{F}$ gives the axis of rotation of the fluid at a point.

3. For a function $f$ we saw the gradient $\nabla f$ is a VF. In fact it's a special kind of VF. Any VF which is the gradient of a function $f$ is *conservative* and the $f$ is a *potential function.*
   There are two facts to note:

   (a) If $\bar{F}$ is conservative then $\nabla \times \bar{F} = \bar{0}$ and consequently if $\nabla \times \bar{F} \neq \bar{0}$ then $\bar{F}$ is not conservative. Moreover if $\nabla \times \bar{F} = \bar{0}$ and $\bar{F}$ is defined for all $(x, y, z)$ then $\bar{F}$ is conservative.

   (b) If we have $\bar{F}$ we can tell if it's conservative by the above method and we can find the potential function too using the iterative method. Make sure to do 2-variable and 3-variable cases.



§15.2 Line Integrals (of Functions and of VFs)

1. If $C$ is a curve and $f$ gives the density at any point then we can define *the line integral of $f$ over/on $C$*, denoted $\int_C f\ ds$, as the total mass of $C$. We evaluate it by parametrizing $C$ as $\bar{r}(t)$ on $[a, b]$ and then $\int_C f\ ds = \int_a^b f(x(t), y(t), z(t))||\bar{r}'(t)||\ dt$. The result is independent of the parametrization and the orientation.
   Sample units: $C$ in cm, $f$ in g/cm and the result in g.

2. If $C$ is the path of an object through a force field $\bar{F}$ then we can define *the line integral of $\bar{F}$ over/on $C$*, denoted $\int_C \bar{F} \cdot d\bar{r}$, as the total work done by $\bar{F}$ as it traverses $C$. The most basic way to evaluate it is by parametrizing $C$ as $\bar{r}(t)$ on $[a, b]$ and then $\int_C \bar{F} \cdot d\bar{r} = \int_a^b \bar{F}(x(t), y(t), z(t)) \cdot \bar{r}'(t)\ dt$.
   Some notes about line integrals of vector fields:

   (a) The orientation (direction) of $C$ matters. If $-C$ is the same curve in the opposite direction then $\int_{-C} \bar{F} \cdot d\bar{r} = -\int_C \bar{F} \cdot d\bar{r}$. This makes sense for work done.

   (b) The parametrization in that direction doesn't matter.

   (c) There is alternate notation for this integral. We can write $\int_C M\ dx + N\ dy + P\ dz$ which means the same as $\int_C (M\,\hat{\imath} + N\,\hat{\jmath} + P\,\hat{k}) \cdot d\bar{r}$. Watch out for things like $\int_C M\ dx$ which looks deceivingly like a regular integral.

   Sample units: $C$ in cm, $\bar{F}$ in $\text{g} \cdot \text{cm/s}$ (dynes) and the result in $\text{g} \cdot \text{cm}^2/\text{s}^2$ (ergs).