Theorem. If there are strictly positive constants c_1 and c_2 such that

$$c_1 d_1(x,y) \le d_2(x,y) \le c_2 d_1(x,y)$$

for all $x, y \in X$, then d_1 and d_2 are topologically equivalent metrics on X.

Proof.

Let U be open in (X, d_1).

Then for $a \in X$ there exists $\epsilon > 0$ such that

$$\{x \in X; d_1(x,a) < \epsilon\} \subset U \ .$$

But then

$$\{x \in X; d_2(x,a) < c_1\epsilon\} \subset \{x \in X; d_1(x,a) < \epsilon\} \subset U$$

and U is open in (X, d_2).

Similarly, if U is open in (X, d_2) it is open in (X, d_1), and the metrics are topologically equivalent.

This criterion is sufficient but not necessary.

Let $X = \mathbb{R}$, and consider the metrics

$$d_1(x,y) = |x-y|$$
$$d_2(x,y) = \frac{|x-y|}{1+|x-y|}$$

Obviously $d_2(x,y) \le d_1(x,y)$ for all x, y, but since $d_1(x,y) = (1+|x-y|)d_2(x,y)$ there is no strictly positive constant c such that $d_2(x,y) \ge c d_1(x,y)$.

On the other hand

$$\{d_1(x,a) < \epsilon\} \subset \{d_2(x,a) < \epsilon\}$$

so that if U is open in $(\mathbb{R}, d_2)$ it is open in $(\mathbb{R}, d_1)$, while if $|x-a| < \epsilon$

$$d_1(x,a) = (1+|x-a|)d_2(x,a)$$
$$\le (1+\epsilon)d_2(x,a)$$

Let $\epsilon_1 = \epsilon/(1+\epsilon)$.

Then $\{d_2(x,a) < \epsilon_1\} \subset \{d_1(x,a) < \epsilon\}$, so that if U is open in $(\mathbb{R}, d_1)$ it is open in $(\mathbb{R}, d_2)$.

For example, in $\mathbb{R}^2$,

$$\begin{aligned} & \max(|x_1-x_2|, |y_1-y_2|) \\ \le \ & |x_1-x_2| + |y_1-y_2| \\ \le \ & 2\max(|x_1-x_2|, |y_1-y_2|) \end{aligned}$$

so that the taxi-cab and sup metrics are equivalent.

In fact all the metrics generated by the norms $||x||_p$ are topologically equivalent on $\mathbb{R}^n$ for finite n.