$q_i$ and eigenvectors $|q_i\rangle$. Then we bra the defining equation of $|q_i\rangle$ through by $\langle q_k|$, and bra the defining equation of $|q_k\rangle$ through by $\langle q_i|$:

$$\langle q_k|Q|q_i\rangle = q_i \langle q_k|q_i\rangle \qquad \langle q_i|Q|q_k\rangle = q_k \langle q_i|q_k\rangle$$

We take the complex conjugate of the second equation from the first. The left-hand side will vanish as $Q$ is hermitian, and so

$$0 = (q_i - q_k^*) \langle q_k|q_j\rangle \tag{1.8}$$

This gives rise to two properties of hermitian operators:

- Letting $k = i$, we find that $q_i = q_i^*$ as $\langle q_i|q_i\rangle \geq 0$. Thus, the eigenvalues are real
- For $k \neq i$, the eigenvalues are distinct. This means that $\langle q_i|q_j\rangle \stackrel{!}{=} 0$, and thus the eigenkets are orthogonal.

As we will see in Quantum Mechanics, the latter property is a logical necessity of the way that we interpret the mathematics of quantum states, as the probability of being in one state *given* another is by definition zero (see A3 notes).

### 1.2.4 Manipulations of Operators

Due to their linearity, the action of operators is sequential:

$$(Q_1Q_2)|\psi\rangle = Q_1(Q_2|\psi\rangle) \neq Q_2(Q_1|\psi\rangle)$$

This inequality has been included to remind readers that in general, operators do not commute. Operators also obey the property that

$$(Q_1Q_2)^\dagger = Q_2^\dagger Q_1^\dagger$$

which can be shown by using the matrix representation for the operators. Using this result, we can prove the case for an arbitrarily large product of operators.



Assume that for the operators $Q_1, Q_2, Q_3, \dots Q_k$ that

$$(Q_1 \dots Q_k)^\dagger = Q_k^\dagger \dots Q_1^\dagger$$

Now, let us examine the case for $n = k + 1$.

$$\begin{aligned}(Q_1, \dots, Q_kQ_{k+1})^\dagger &= [(Q_1 \dots Q_k)(Q_{k+1})]^\dagger \\ &= (Q_{k+1})^\dagger (Q_1 \dots Q_k)^\dagger \\ &= Q_{k+1}^\dagger Q_k^\dagger \dots Q_1^\dagger\end{aligned}$$

using the inductive assumption. Thus, it follows from the principle of mathematical induction that

$$(Q_1Q_2 \dots Q_n)^\dagger = Q_n^\dagger Q_{n-1}^\dagger \dots Q_1^\dagger \tag{1.9}$$

is true for all $n \geq 0$

## 2.3 Fourier Transforms

The *Fourier transform* of a function is the extension of this kind of Fourier series to an infinite domain. Let us suppose that the function can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2in\pi}{L}x}$$

where the coefficients $c_n$ may be complex (allowing us to define this expansion for both real and complex functions). What are the coefficients $c_n$?

$$\begin{aligned} c_n &= \frac{2}{|b-a|}\int_a^b dx\ \left(e^{\frac{2in\pi}{L}x}\right)^* f(x) \\ &= \frac{2}{|b-a|}\int_a^b dx\ e^{-\frac{2in\pi}{L}x} f(x) \end{aligned}$$

Let $k_n = \frac{2\pi n}{L}$. Then, we can write the basis elements as

$$\begin{aligned} e_n(x) &= \frac{\sqrt{2\pi}}{L} e^{\frac{2in\pi}{L}x} \\ &= \frac{1}{\sqrt{2\pi}}\frac{2\pi}{L} e^{ik_n x} \\ &= \frac{1}{\sqrt{2\pi}}\left(k_{n+1}-k_n\right) e^{ik_n x} \end{aligned}$$

Let us define the entity

$$\tilde{f}(k_n) = \frac{1}{\sqrt{2\pi}}\int_a^b dx\ e^{-ik_n x} f(x)$$

We can then write that

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} e_n(x)\,\tilde{f}(k_n) \\ &= \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} e^{ik_n x}\left(k_{n+1}-k_n\right)\,\tilde{f}(k_n) \end{aligned}$$



Taking the limit as $[a,b] \to [-\infty,\infty]$, $k_{n+1} \approx k_n$, and so the sum above becomes an integral. This means that we can define the Fourier transform (often abbreviated as FT) of a function as

$$\boxed{\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\ e^{-ikx}\ f(x)} \tag{2.7}$$

Evidently, this is an invertible process, and so we defined the *Inverse Fourier transform* as

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ e^{ikx}\ \tilde{f}(k)} \tag{2.8}$$

The uses and consequences of these transforms will become more relevant when used in quantum mechanics, as well as for Fourier Optics (see A2 notes for more details).

One particularly important Fourier transform to consider is that of the Gaussian distribution due to its relevance to many parts of Physics involving probability distributions. Consider a gaussian of the form

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

Then, it's Fourier transform is given by

$$\begin{aligned}
\widetilde{g}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \; e^{-ikx} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \\
&= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} dx \; e^{-\left(\frac{x^2}{2\sigma^2} + ikx\right)}
\end{aligned}$$

To solve this integral, we need to complete the square on the exponent.

$$\begin{aligned}
\frac{x^2}{2\sigma^2} + ikx &= \frac{1}{2\sigma^2}\left(x^2 + 2ik\sigma^2 x\right) \\
&= \frac{1}{2\sigma^2}\left((x + ik\sigma^2)^2 - (ik\sigma^2)^2\right) \\
&= \frac{1}{2\sigma^2}\left((x + ik\sigma^2)^2 + (k\sigma^2)^2\right)
\end{aligned}$$

Substituting this in:

$$\begin{aligned}
\widetilde{g}(k) &= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} dx \; e^{-\frac{1}{2\sigma^2}\left((x+ik\sigma^2)^2 + (k\sigma^2)^2\right)} \\
&= \frac{1}{2\pi\sigma} \; e^{-\frac{1}{2}(k\sigma)^2} \int_{-\infty}^{\infty} dx \; e^{-\frac{1}{2\sigma^2}\left((x+ik\sigma^2)^2\right)} \\
&= \frac{1}{2\pi\sigma} \; e^{-\frac{1}{2}(k\sigma)^2} \int_{-\infty+ik\sigma^2}^{\infty+ik\sigma^2} du \; e^{-\frac{1}{2\sigma^2}u^2} \\
&= \frac{1}{2\pi\sigma} \; e^{-\frac{1}{2}(k\sigma)^2} \sqrt{2\pi}\sigma \\
\rightarrow \widetilde{g}(k) &= \frac{1}{\sqrt{2\pi}} \; e^{-\frac{1}{2}(k\sigma)^2}
\end{aligned}$$



Thus, the Fourier transform of a gaussian of dispersion $\sigma$ is $\sigma\times$ the gaussian of dispersion $1/\sigma$ in $k$-space. Thus, if a function is constrained in $x$-space, then it is spread out in $k$-space. For example, if we have a properly-normalised wave packet that is very bunched in $x$-space, we require a large variation in wave-number $k$ in order to create it; this means that the distribution in $k$ space must be much wider and more spread out.

### 2.3.1 Properties of the Fourier Transform

The FT of a given function has some quite useful properties, as follows:

- Rescaling: $f(ax) = \frac{1}{a} f(k/a)$
- Shifting: $f(a+x) = e^{ika} \, \widetilde{f}(k)$
- Exponential Multiplication: $e^{iqx} \, f(x) = \widetilde{f}(k-q)$
- Derivative: $\frac{\partial f}{\partial x} = ik\widetilde{f}(k)$
- Polynomial Multiplication: $xf(x) = i\frac{\partial \widetilde{f}}{\partial k}$

## 2.4 The Dirac Delta Function

The Dirac Delta function is defined by the following properties:

$$\delta(x-a)=0 \tag{2.11}$$

$$\int_{-\infty}^{\infty} dx \;\; \delta(x-a)=0 \tag{2.12}$$

$$\int_{-\infty}^{\infty} dx \;\; f(x)\delta(x-a)=f(a) \tag{2.13}$$

It can be thought of as an infinite 'spike' at the point $x=a$ of zero width and area one. Evidently, no function exists that actually satisfies these properties. We can instead think about $\delta(x)$ as being as the limit for $\varepsilon\to 0$ of a function $\delta_\varepsilon(x)$ for which

$$\lim_{\varepsilon\to 0}\int dx \;\; f(x)\delta(x-a)=f(a)$$

The most important of these is the Gaussian form, defined as

$$\boxed{\delta_\varepsilon(x-a)=\frac{1}{\sqrt{2\pi\varepsilon}}e^{-\frac{(x-a)^2}{2\varepsilon^2}}} \tag{2.14}$$

Graphically, we can think of this as 'squashing' down a gaussian of finite width into this infinite 'spike', as illustrated below.

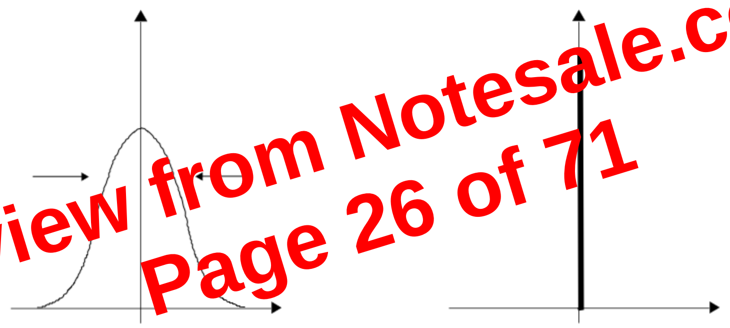



Figure 2.3: The limiting case of a Gaussian with dispersion $\varepsilon$ (left) to the Dirac Delta function (right)

Let us now demonstrate that (2.14) does indeed satisfy the required properties.

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} dx \;\; f(x)\,\delta_\varepsilon(x-a)\\
&=\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} dx \;\; f(x)\,\frac{1}{\sqrt{2\pi\varepsilon}}e^{-\frac{(x-a)^2}{2\varepsilon^2}}\\
&=\lim_{\varepsilon\to 0}\frac{1}{\sqrt{2\pi\varepsilon}}\int_{-\infty}^{\infty} dx \;\; f(x)\,e^{-\frac{(x-a)^2}{2\varepsilon^2}}\\
&=\lim_{\varepsilon\to 0}\frac{1}{\sqrt{2\pi\varepsilon}}\int_{-\infty}^{\infty} dx \;\; e^{-\frac{(x-a)^2}{2\varepsilon^2}}\left[f(a)+(x-a)f'(a)+\frac{1}{2}(x-a)^2f''(a)+\dots\right]\\
&=\lim_{\varepsilon\to 0}\frac{1}{\sqrt{2\pi\varepsilon}}\left[f(a)\int_{-\infty}^{\infty} dx \;\; e^{-\frac{(x-a)^2}{2\varepsilon^2}}+f'(a)\int_{-\infty}^{\infty} dx \;\; (x-a)\,e^{-\frac{(x-a)^2}{2\varepsilon^2}}+\dots\right]\\
&=\lim_{\varepsilon\to 0}\left[f(a)+\frac{1}{2}f''(a)\,\varepsilon^2+\mathcal{O}(\varepsilon^4)\right]\\
&=f(a)
\end{aligned}$$

# 3. *Sturm-Liouville Operators*

This section will define, and cover the basic concepts of, Sturm-Liouville Operators, including:

- The Eigenvalue Problem
- Definition and Properties
- Methods of Solution
- Known Differential Equations
- Some Quantum Mechanics

The concept of the Sturm-Liouville operator plays a very large role in much of Physics as it allows us to solve many second order differential equations, the most common type of differential equation that we encounter. This means that once again, it is imperative that students are very familiar with this crucial piece of mathematical apparatus that we can use to tackle proper Physics. Note that the "•" symbol is used to show where the function is placed when the operator is applied.

where in the last step we have used the substitution that $x = \lambda \bar{x}$. The solutions to this equation are Bessel functions. Very generally, they can be defined by the generating function

$$\boxed{G(x,t) = \exp\left[\frac{1}{2}x\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n} \tag{3.21}$$

Graphically, they are represented as

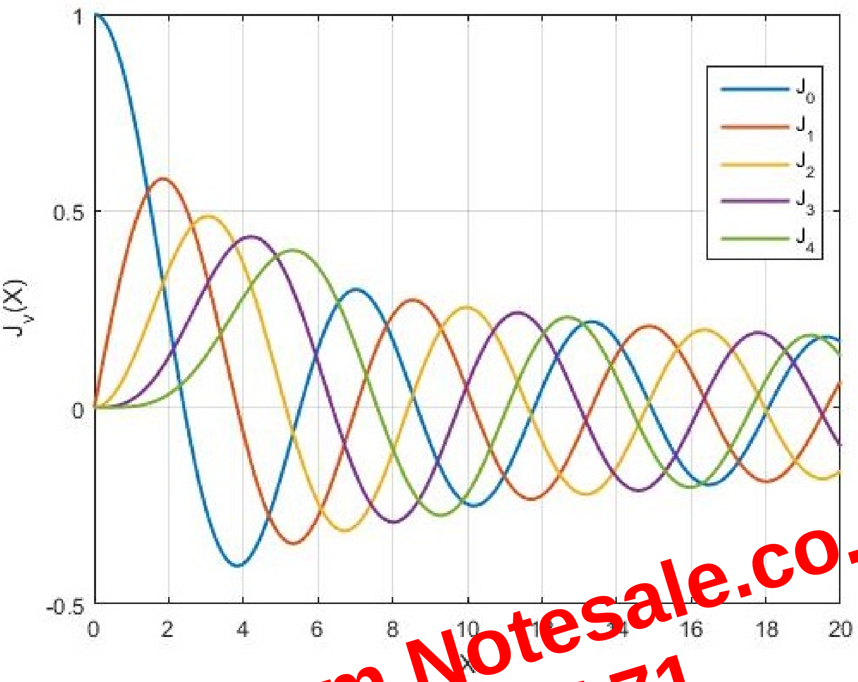


Figure 3.3: The first five Bessel functions of the first kind



As can be seen from the Figure below, Bessel functions are oscillatory, and very slowly decaying (there is very little difference in amplitude between two adjacent peaks). The roots of the functions are particularly important for solving boundary value problems

This is the case for the general solution above in two dimensions.

$$F(x,t) = \sum_{n=1}^{\infty} \sin(k_n x)\left(a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\right)$$

for $k_n = \pi n/L$ and $\omega_n = \pi n c/L$. At $t = 0$, we have that

$$F = \sum_{n=1}^{\infty} a_n \sin\left(\frac{\pi n x}{L}\right)$$
$$\dot{F} = \sum_{n=1}^{\infty} \omega_n b_n \sin\left(\frac{\pi n x}{L}\right)$$

We can then find the Fourier coefficients by the normal method. In this case, it is evident that all $b_n = 0$. Finding $a_n$:

$$\begin{aligned}
a_n &= \frac{2}{L}\int_0^{\infty} dx\ \ 2A\sin\left(\frac{3\pi x}{2L}\right)\cos\left(\frac{\pi x}{2L}\right)\sin\left(\frac{\pi n x}{L}\right) \\
&= \frac{4A}{L}\int_0^{\infty} dx\ \ \frac{1}{2}\left[\sin\left(\frac{\pi x}{L}\right) + \sin\left(\frac{2\pi x}{L}\right)\right]\sin\left(\frac{\pi n x}{L}\right) \\
&= \frac{2A}{L}\int_0^{\infty} dx\ \ \sin^2\left(\frac{\pi x}{L}\right)\delta_{n1} + \sin^2\left(\frac{2\pi x}{L}\right)\delta_{n2} \\
&= A\left(\delta_{n1} + \delta_{n2}\right)
\end{aligned}$$

Here we have used the fact that $\sin x$, $\sin 2x$, $\ldots \sin nx$ form an orthonormal basis. Thus, the final displacement is given by

$$F(x,t) = A\left(\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi c t}{L}\right) + \sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi c t}{L}\right)\right)$$

As expected, the subsequent solution is simply the superposition of the time evolution of the initially excited modes, as we saw in CP4 last year.



### 4.3.2 The Heat Diffusion Equation

The heat diffusion equation, as the name suggests, describes the flow of heat/temperature in a given medium.

$$\boxed{\frac{\partial T}{\partial t} = \alpha \nabla^2 T} \tag{4.5}$$

We will learn how to derive this equation in the notes on Kinetic Theory. There will also be detailed solutions to harder problems involving heat sources in said notes.

*A one dimensional bar extends from $x = 0$ to $x = L$, and is perfectly insulated at $x = L$. At $t < 0$ the temperature throughout the bar is $0K$, and at $t = 0$ the end at $x = 0$ is placed in thermal contact with a heat bath at temperature $T_o$. Estimate the time taken for the average temperature of the bar to reach $T = 0.9T_o$.*

Again, we will use a substitution like that outlined in Section 4.2 by letting $T(x,t) = f(t)g(x)$.

$$g\frac{df}{dt} = \alpha f \frac{d^2 g}{dx^2}$$
$$\frac{1}{\alpha f}\frac{df}{dt} = \frac{1}{g}\frac{d^2 g}{dx^2}$$

## 4.4 Green's Functions

In all of the above cases, we have been solving the homogeneous case of (4.1). We can solve this by solving the similar equation of

$$\hat{D}\, G(x,t) = \delta(x-t)$$

Note that both sides of this equation are transnationally invariant, such that $G = G(x-t)$. The solution to the equation is then given by

$$\boxed{\phi(x) = \int G(x,t) f(t)\, dt} \tag{4.11}$$

Essentially, we can think of $f(x)$ as the 'source', and that the result that we obtain is the convolution of $G(x,t)$ and the source. If we observe the fact that the Dirac delta function is the identity operator in function space, it becomes clear that the function $G(x,t)$ is essentially the inverse of $\hat{D}$. This means that *we require that $\hat{D}$ is of Sturm-Liouville form.*

Suppose that $f(x)$ is come linear combination of the solutions $\phi_n(x)$.

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

Now, we will suppose further that $G(x,t)$ is of the form:

$$\boxed{G(x,t) = \sum_{n=0}^{\infty} \frac{\phi_n^*(t)\phi_n(x)}{\lambda_n}} \tag{4.12}$$



Let us substitute this into the defining equation:

$$\begin{aligned}
\hat{D}\, G(x,t) &= \hat{D} \sum_{n=0}^{\infty} \frac{\phi_n^*(t)\phi_n(x)}{\lambda_n} \\
&= \sum_{n=0}^{\infty} \frac{\phi_n^*(t)\, \hat{D}\, \phi_n(x)}{\lambda_n} \\
&= \sum_{n=0}^{\infty} \phi_n^*(t)\phi_n(x)
\end{aligned}$$

We have used the fact that $\phi_n(x)$ is an eigenfunction of the operator $\hat{D}$. Recall that

$$`f(x) = \int dt\, \delta(x-t) f(t)$$

Multiplying both sides by $f(t)$ and integrating

$$\begin{aligned}
\int dt\, \hat{D}\, G(x,t) f(t) &= \int dt \sum_{n=0}^{\infty} \phi_n^*(t)\phi_n(x)\, f(t) \\
&= \sum_{n=0}^{\infty} \phi_n(x) \int dt\, \phi_n^*(t) \sum_{m=0}^{\infty} c_m \phi_m(t) \\
&= \sum_{n=0}^{\infty} \phi_n(x) c_m \delta_{mn} \\
&= \sum_{n=0}^{\infty} c_n \phi_n(x) \\
&= f(x)
\end{aligned}$$

# 5. *Probability and Statistics*

This chapter aims to cover the basic probability concepts required to tackle probabilistic Physics, including:

- Basic Probability Concepts
- Some Probability Distributions
- Basic Error Propagation

This is by no means a very comprehensive treatment of probability and statistics, more the bare-minimum to get by. If you wish for a more detailed analysis, we would recommend consulting a proper textbook. Note that throughout much of this chapter we will be using $x$ to refer to some general variable; it does not explicitly refer to displacement in this case.

Here we have split up the sums so then we can more easily introduce some of the quantities that we know, such as the mean. Be careful to the number of terms that come out of the summations; for example, the two sums will give $n$ and $n-1$ respectively.

$$\begin{aligned}
\widetilde{\sigma}^2 &= \sum_i \frac{1}{n}\left(\left(1-\frac{2}{n}\right)\left\langle x_i^2\right\rangle - \frac{2}{n}\sum_{i\neq j}\left\langle x_i x_j\right\rangle + \frac{1}{n^2}\sum_j\left\langle x_j^2\right\rangle + \frac{1}{n^2}\sum_j\sum_{k\neq j}\left\langle x_j x_k\right\rangle\right)\\
&= \frac{1}{n}\sum_i\left(\left(1-\frac{2}{n}\right)\left(\sigma^2+\mu^2\right) - 2\left(1-\frac{1}{n}\right)\mu^2 + \frac{1}{n}\left(\sigma^2+\mu^2\right) + \left(1-\frac{1}{n}\right)\mu^2\right)\\
&= \left(1-\frac{2}{n}\right)\left(\sigma^2+\mu^2\right) - 2\left(1-\frac{1}{n}\right)\mu^2 + \frac{1}{n}\left(\sigma^2+\mu^2\right) + \left(1-\frac{1}{n}\right)\mu^2\\
&= \left(1-\frac{1}{n}\right)\sigma^2
\end{aligned}$$

In this case, we have found that the estimator is actually based. To correct this, the estimator for the variance is defined as

$$\boxed{\widetilde{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i-\mu\right)^2} \tag{5.8}$$

This is the case as there are only $n-1$ independent data-points that you can use to measure the variance, not $n$. It is meaningless to talk about the variance of a single data-point.

The simple re-labelling of summation indices $l = k - 1$ and $m = n - 1$ has been used. Thus, the mean of the binomial distribution is given by

$$\boxed{\mu = np} \tag{5.10}$$

Now for the variance. We first need to calculate $\langle x^2 \rangle$.

$$\begin{aligned}
\langle x^2 \rangle &= \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} \\
&= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)} \\
&= np \sum_{l=0}^{m} (l+1) \binom{m}{l} p^l (1-p)^{m-l} \\
&= np \left( (n-1)p \sum_{l=1}^{n} \binom{m-1}{l-1} p^{l-1} (1-p)^{(m-1)-(j-1)} + \sum_{l=0}^{m} \binom{m}{l} p^l (1-p)^{m-l} \right) \\
&= np \left( (n-1)p(p + (1-p))^{m-1} + (p + (1-p))^m \right) \\
&= n^2 p^2 + np(1-p)
\end{aligned}$$

Again, we have used the same trick with re-labelling indices.

$$\begin{aligned}
\sigma^2 &= \langle x^2 \rangle - \mu^2 \\
&= n^2 p^2 + np(1-p) - (np)^2 \\
&= np(1-p)
\end{aligned}$$

Thus, the variance of the Binomial distribution is given by

$$\boxed{\sigma^2 = np(1-p)} \tag{5.11}$$



### 5.2.2 Poisson

Consider the limiting case of the Binomial distribution in which $n$ depends on some continuous variable (such as time), and can be arbitrary small. In this case, we define the mean by

$$\mu = \lim_{n \to \infty} np$$

Now, substitute this into (5.9) and simplify using this large parameter $n$.

$$\begin{aligned}
f(k) &= \lim_{n \to \infty} \frac{n!}{(n-k)!k!} \left(\frac{\mu}{n}\right)^k \left(1 - \frac{\mu}{n}\right)^{n-k} \\
&= \frac{\mu^k}{k!} \lim_{n \to \infty} \underbrace{\frac{n!}{(n-k)} \left(\frac{1}{n}\right)^k}_{(1)} \underbrace{\left(1 - \frac{\mu}{n}\right)^k}_{(2)} \underbrace{\left(1 - \frac{\mu}{n}\right)^n}_{(3)}
\end{aligned}$$