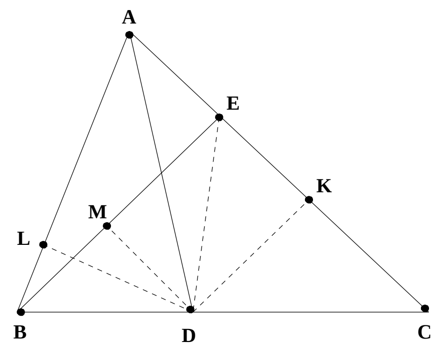
Solutions to CRMO-2007 Problems

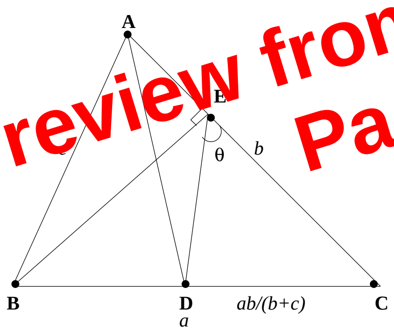
1. Let ABC be an acute-angled triangle; AD be the bisector of $\angle BAC$ with D on BC; and BE be the altitude from B on AC. Show that $\angle CED > 45°$.

Solution:

Draw DL perpendicular to AB; DK perpendicular to AC; and DM perpendicular to BE. Then $EM = DK$. Since AD bisects $\angle A$, we observe that $\angle BAD = \angle KAD$. Thus in triangles ALD and AKD, we see that $\angle LAD = \angle KAD$; $\angle AKD = 90° = \angle ALD$; and AD is common. Hence triangles ALD and AKD are congruent, giving $DL = DK$. But $DL > DM$, since BE lies inside the triangle(by acuteness property). Thus $EM > DM$. This implies that $\angle EDM > \angle DEM = 90° - \angle EDM$. We conclude that $\angle EDM > 45°$. Since $\angle CED = \angle EDM$, the result follows.



Alternate Solution:



Let $\angle CED = \theta$. We have $CD = ab/(b+c)$ and $CE = a\cos C$. Using sine rule in triangle CED, we have

$$\frac{CD}{\sin\theta} = \frac{CE}{\sin(C+\theta)}.$$

This reduces to

$$(b+c)\sin\theta\cos C = b\sin C\cos\theta + b\cos C\sin\theta.$$

Simplification gives $c\sin\theta\cos C = b\sin C\cos\theta$ d so that

$$\tan\theta = \frac{b\sin C}{c\cos C} = \frac{\sin B}{\cos C} = \frac{\sin B}{\sin(\pi/2 - C)}.$$

Since ABC is acute-angled, we have $A < \pi/2$. Hence $B + C > \pi/2$ or $B > (\pi/2) - C$. Therefore $\sin B > \sin(\pi/2 - C)$. This implies that $\tan\theta > 1$ and hence $\theta > \pi/4$.

2. Let a, b, c be three natural numbers such that $a < b < c$ and $\gcd(c-a, c-b) = 1$. Suppose there exists an integer d such that $a+d, b+d, c+d$ form the sides of a right-angled triangle. Prove that there exist integers l, m such that $c + d = l^2 + m^2$.

Solution:

We have

$$(c+d)^2 = (a+d)^2 + (b+d)^2.$$

This reduces to

$$d^2 + 2d(a+b-c) + a^2 + b^2 - c^2 = 0.$$