2) If $Y = l$, then the conditional distribution of $X|(Y = l)$ is $Binomal(n-l, \frac{p}{1-\theta})$.
Proof.

$$\begin{aligned} P(X=k|Y=l) &= \frac{P(X=k,Y=l)}{P(Y=l)} = \frac{\frac{n!}{k!l!(n-k-l)!}p^k\theta^l(1-p-\theta)^{n-k-l}}{\frac{n!}{l!(n-l)!}\theta^l(1-\theta)^{n-l}} \\ &= \binom{n-l}{k}\left(\frac{p}{1-\theta}\right)^k\left(1-\frac{p}{1-\theta}\right)^{n-l-k} \end{aligned}$$

for $x = 0, 1, ..., (n-y)$. Hence $(X|Y=y) \sim Binomial\left(n-y, \frac{p}{1-\theta}\right)$. $\square$

This is intuitively obvious. Consider those trials for which "failure" (or 0) did not occur. There are $(n-l)$ such trials, for each of which the probability that 1 occurs is actually the conditional probability of 1 given that 0 has not occurred, i.e. $\frac{p}{1-\theta}$. So you have the standard binomial set-up.

3) We shall now use the results on conditional distributions (Notes 5) and the above properties to find $Cov(X,Y)$ and the coefficient of correlation $\rho(X,Y)$.

We proved that $E[XY] = E[YE[X|Y]]$ (see the last page of Notes 5). According to property 2), $E[X|Y=l] = (n-l)\frac{p}{1-\theta}$ and thus $E[X|Y] = (n-Y)\frac{p}{1-\theta}$. Hence

$$\begin{aligned} E[XY] &= E\left[Y \times (n-Y)\frac{p}{(1-\theta)}\right] = \frac{p}{1-\theta}E(nY-Y^2) = \frac{p}{1-\theta}\left(n^2\theta - n\theta(1-\theta) - n^2\theta^2\right) \\ &= \frac{p}{(1-\theta)}[n(n-1)\theta(1-\theta)] = n(n-1)p\theta \end{aligned}$$

Therefore $Cov(X,Y) = E[XY] - E[X]E[Y] = n(n-1)p\theta - n^2p\theta = -np\theta$ and hence

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{-np\theta}{\sqrt{n^2p(1-p)\theta(1-\theta)}} = -\left(\frac{p\theta}{(1-p)(1-\theta)}\right)^{\frac{1}{2}}$$

Note that if $p + \theta = 1$ then $Y = n - X$ and there is an exact linear relation between Y and X. In this case it is easily seen that $\rho(X,Y) = -1$.

Definition of the multinomial distribution

Now suppose that there are k outcomes possible at each of the n independent trials. Denote the outcomes $A_1, A_2, ..., A_k$ and the corresponding probabilities $p_1, ..., p_k$ where $\sum_{j=1}^{k} p_j = 1$. Let X_j count the number of times A_j occurs. Then

$$P(X_1 = x_1, ..., X_{k-1} = x_{k-1}) = \frac{n!}{x_1!x_2!...x_{k-1}!(n-\sum_{j=1}^{k-1}x_j)!}p_1^{x_1}p_2^{x_2}...p_{k-1}^{x_{k-1}}p_k^{n-\sum_{j=1}^{k-1}x_j}$$

where $x_1, x_2, ..., x_{k-1}$ are non-negative integers with $\sum_{j=1}^{k-1} x_j \leq n$.