Advanced Calculus - Integration

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Page 1 of 15

1.5.1 Example 5

We can show that $\int_a^x (t+1)\,dt = \frac{1}{2}(x^2 - a^2) + (x - a)$. This means that $A(x) = \int_a^x (t+1)\,dt = \frac{1}{2}(x^2 - a^2) + (x - a)$. Differentiating $A(x)$ we get,

$$A'(x) = \frac{1}{2}(2)(x) + 1 = x + 1 = f(x)$$

1.5.2 Example 6

Let $f(x) = 3x^2$. We observe that if $F(x) = x^3$ then $F'(x) = 3x^2 = f(x)$. Part (2) of the fundemental theorem of Calculus states,

$$\int_a^b 3x^2\,dx = F(b) - F(a) = b^3 - a^3$$

1.6 Proof of the fundemental theorem of Calculus

If f is a continuous function and F is any function such that $F'(x) = f(x)$, then F is said to be an antiderivative of f. F is also known as the primitive integral or the indefinite integral.

1.6.1 Part 1

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

$$A(x+h) - A(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt$$

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Page 8 of 15

From the mean value theorem we have,

$$\frac{1}{x+h-x}\int_x^{x+h} f(t)\,dt = f(c) = \frac{1}{h}\int_x^{x+h} f(t)\,dt \implies hf(c) = \int_x^{x+h} f(t)\,dt$$

For some $c = c_h \in (x, x+h)$, meaning $x < c_h < x+h$.

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(h)}{h} = \lim_{h \to 0} f(c) = \lim_{h \to 0} f(c_h),$$

With $x < c_h < x + h$. By the sandwich theorem as $h \to 0$, $c_h \to x$, so $A'(x) = f(x)$.

1.6.2 Part 2

If $F'(x) = f(x)$ then, $\int_a^b f(x)\,dx = F(b) - F(a)$. From part (1), $A(x) = \int_a^x f(t)\,dt$ satisfies $A'(x) = f(x)$. Introducing the equation $G(x) = F(x) - A(x) \implies G'(x) = F'(x) - A'(x) \implies G(x) = f(x) - f(x) = 0$. So G is a constant, $G(x) = c$. So $F(x) = A(x) + G(x) \implies F(x) = A(x) + c$. So,

$$F(x) = \int_a^x f(t)\,dt + c$$

From the original statement,

$$F(b) - F(a) = \left(\int_a^b f(t)\,dt + c\right) - \left(\int_a^a f(t)\,dt + c\right) = \int_a^b f(t)\,dt$$

as $\int_a^a f(t)\,dt = 0$ from integral property (5).

To conclude, the fundemental theorem of Caculus says (1) differentiation is the inverse of integration. (2) integration is the inverse of differentiation.