can write it in product notation in different forms:

$$\prod_{k=m}^{n} k,$$
$$\prod_{m \le k \le n} k,$$
$$\prod_{k \in S} k, \text{ where } S = \{\, m, m+1, \ldots, n \,\}.$$

It may happen that the sum or product should be evaluated on the empty set. By definition, in such situations the sum is always 0 and the product is always 1, e.g.

$$\sum_{k \in \emptyset} k = 0,$$
$$\prod_{k \in \emptyset} k = 1.$$

If S and T be two disjoint sets, then

$$\sum_{k \in S} k + \sum_{k \in T} k = \sum_{k \in S \cup T} k,$$
$$\prod_{k \in S} k \cdot \prod_{k \in T} k = \prod_{k \in S \cup T} k.$$

Note, that this is true even if S or T is the empty set. (This is the main reason we define the empty sum to be 0 and the empty product to be 1.)

There is a special notation for the product of positive integers up to n, that is, when we multiply the elements of

$$S_n = \{\, k \mid k \text{ is a positive integer}, k \le n \,\} = \{\, 1, 2, \ldots, n \,\}.$$

The product of the elements of S_n is called n *factorial* and denoted by $n!$, that is,

$$n! = \prod_{k \in S_n} k = \prod_{k=1}^{n} k = 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n.$$

We even define $0!$, that is, the products of elements of S_0:

$$0! = \prod_{k \in S_0} k = \prod_{k \in \emptyset} k = 1.$$

Factorials are always computed before any other operation. For example

$$2 + 3! = 2 + 1 \cdot 2 \cdot 3 = 2 + 6 = 8,$$
$$(2 + 3)! = 5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120.$$

Exercise 1.9. Expand the following sums.

(a) $\sum_{i=4}^{7} i$,
(b) $\sum_{i=1}^{5} (i^2 - i)$,
(c) $\sum_{i=1}^{4} 10^i$,
(d) $\sum_{2 \le i \le 5} \frac{1}{2^i}$,
(e) $\sum_{i \in S} (-1)^i$, where $S = \{ 2, 3, 5, 8 \}$.

Exercise 1.10. Write the following expressions in summation notation.

(a) $2 + 4 + 6 + 8 + 10$,
(b) $1 + 4 + 7 + 10$,
(c) $\frac{1}{4} + \frac{1}{2} + 1 + 2 + 4$,
(d) $\frac{1}{4} - \frac{1}{2} + 1 - 2 + 4$.

Exercise 1.11. Expand the following products.

(a) $\prod_{i=-4}^{-1} i$,
(b) $\prod_{i=1}^{4} (i^2)$,
(c) $\prod_{i=1}^{3} 2^i$,
(d) $\prod_{-2 \le i \le 3} \frac{1}{2^i}$,
(e) $\prod_{i \in S} (-1)^i$, where $S = \{ 2, 4, 6, 7 \}$.

Exercise 1.12. Write the following expressions in product notation.

(a) $1 \cdot 3 \cdot 5 \cdot 7$,
(b) $(-1) \cdot 2 \cdot 5 \cdot 8$,
(c) $\frac{1}{9} \cdot \frac{1}{3} \cdot 1 \cdot 3 \cdot 9$.

Exercise 1.13. Compute the values of $n!$ for every $n \in \{ 0, 1, 2, 3, 4, 5, 6, 7, 8 \}$.

a remainder r_1. Then we apply the Division algorithm for b and r_1 to get a new quotient q_2 and a new remainder r_2. We continue, we divide r_1 by r_2 to obtain q_3 and r_3. We stop if we obtain a zero remainder. Since the procedure produces a decreasing sequence of non-negative integers so must eventually terminate by descent. The last non-zero remainder is the greatest common divisor of a and b.

As an example we compute $\gcd(553, 161)$. We write the computations in the following way:

$$\begin{aligned} 553 &= 3 \cdot 161 + 70 \quad q_1 = 3, r_1 = 70 \\ 161 &= 2 \cdot 70 + 21 \quad q_2 = 2, r_2 = 21 \\ 70 &= 3 \cdot 21 + 7 \quad q_3 = 3, r_3 = 7 \\ 21 &= 3 \cdot 7 + 0 \quad q_4 = 3, r_4 = 0. \end{aligned}$$

That is, the last non-zero remainder is 7, so $\gcd(553, 161) = 7$. If we would like to express 7 as $553x + 161y$ for some $x, y \in \mathbb{Z}$, we can do it by working backwards

$$\begin{aligned} 7 &= 70 - 3 \cdot 21 \\ &= 70 - 3 \cdot (161 - 2 \cdot 70) = -3 \cdot 161 + 7 \cdot 70 \\ &= -3 \cdot 161 + 7 \cdot (553 - 3 \cdot 161) = 7 \cdot 553 - 24 \cdot 161. \end{aligned}$$

It follows that $x = 7$ and $y = -24$.

Exercise 1.16. Use the Euclidean algorithm to find $\gcd(a, b)$ and compute integers x and y for which

$$ax + by = \gcd(a, b):$$

(a) $a = 678, b = 567$,
(b) $a = 803, b = 319$,
(c) $a = 2701, b = 2257$,
(d) $a = 3397, b = 1849$.

Now, rewrite 250_{10} into base 3:

$$\begin{aligned}
250 &= 83 \cdot 3 + 1,\\
83 &= 27 \cdot 3 + 2,\\
27 &= 9 \cdot 3 + 0,\\
9 &= 3 \cdot 3 + 0,\\
3 &= 1 \cdot 3 + 0,\\
1 &= 0 \cdot 3 + 1.
\end{aligned}$$

The remainders backwards are 1, 0, 0, 0, 2, 1, thus

$$372_8 = 250_{10} = 100021_3.$$

Finally, we mention that some rewriting can be done much quicker if one base is a full power of another. For example, $8 = 2^3$, and then every base 8 digit can be rewritten easily to three base 2 digits:

$$\begin{aligned}
0_8 &= 000_2, & 1_8 &= 001_2,\\
2_8 &= 010_2, & 3_8 &= 011_2,\\
4_8 &= 100_2, & 5_8 &= 101_2,\\
6_8 &= 110_2, & 7_8 &= 111_2.
\end{aligned}$$

Going from right to left, every three base 2 digits can be easily rewritten into base 8, as well. Thus, it is easy to rewrite 372_8 into base 2 or 10101_2 into base 8:

$$\begin{aligned}
372_8 &= 011\ 111\ 010_2 = 11111010_2,\\
10101_2 &= 010\ 101_2 = 25_8.
\end{aligned}$$

Similarly, as $16 = 2^4$, every base 16 digit can be rewritten easily to four base 2 digits:

$$
\begin{aligned}
0_{16} &= 0000_2, & 1_{16} &= 0001_2,\\
2_{16} &= 0010_2, & 3_{16} &= 0011_2,\\
4_{16} &= 0100_2, & 5_{16} &= 0101_2,\\
6_{16} &= 0110_2, & 7_{16} &= 0111_2,\\
8_{16} &= 1000_2, & 9_{16} &= 1001_2,\\
A_{16} &= 1010_2, & B_{16} &= 1011_2,\\
C_{16} &= 1100_2, & D_{16} &= 1101_2,\\
E_{16} &= 1110_2, & F_{16} &= 1111_2.
\end{aligned}
$$

Going from right to left, every four base 2 digits can be easily rewritten into base 16, as well. Thus, it is easy to rewrite AFE_{16} into base 2 or 10101_2 into base 16:

$$
\begin{aligned}
AFE_{16} &= 1010\ 1111\ 1110_2 = 101011111110_2,\\
10101_2 &= 0001\ 0101_2 = 15_{16}.
\end{aligned}
$$

We have to stress, though, that this method only works *if one base is a full power of the other.* Finally, base 8 numbers can be easily changed to base 16 (and vice versa) by first changing them to base 2, and then into the other base:

$$
\begin{aligned}
372_8 &= 011\ 111\ 010_2 = 11111010_2 = 1111\ 1010_2 = FA_{16},\\
AFE_{16} &= 1010\ 1111\ 1110_2 = 101011111110_2 = 101\ 011\ 111\ 110_2 = 5376_8.
\end{aligned}
$$

Exercise 1.19. (a) Write the following numbers into base 10: 111001101_2, 1010101_2, 11111_2, 10110_2, 101010101_2, 10001000_2, 1010111_2, 111101_2, 21102_3, 1234_5, 1234_7, 1234_8, 777_8, 345_8, 2012_8, 4565_8, 1123_8, 666_8, 741_8, CAB_{16}, BEE_{16}, EEE_{16}, $4D4_{16}$, ABC_{16}, $9B5_{16}$, DDD_{16}, $3F2_{16}$.

(b) Write the following decimal numbers into base 2, 3, 5, 7, 8, 9, 16: 64_{10}, 50_{10}, 16_{10}, 100_{10}, 2012_{10}, 200_{10}, 151_{10}, 48_{10}, 99_{10}, 999_{10}.

After solving Exercise 2.11, one suspects that the number of subsets depend only on the cardinality of the set, and not on the actual elements of the set. This is true in general: for example if a set has three elements, then we might as well name the elements a, b and c, and then its subsets will be exactly the same as we determined in Exercise 2.11.

Let us try to determine the number of subsets of a set with given cardinality. Let S be a set of cardinality 0, i.e. $S = \emptyset$. Then S has only one subset: $\emptyset$. If S is a set of cardinality 1, e.g. $S = \{ a \}$, then it has two subsets: $\{ \} = \emptyset$, $\{ a \} = S$. If S is a set of cardinality 2, e.g. $S = \{ a, b \}$, then it has four subsets: $\{ \} = \emptyset$, $\{ a \}$, $\{ b \}$, $\{ a, b \} = S$. If S is a set of cardinality 3, e.g. $S = \{ a, b, c \}$, then it has eight subsets: $\{ \} = \emptyset$, $\{ a \}$, $\{ b \}$, $\{ c \}$, $\{ a, b \}$, $\{ a, c \}$, $\{ b, c \}$, $\{ a, b, c \} = S$. Figure 2.1 shows all subsets of $\{ a, b, c \}$. In this figure, two sets are connected if the lower one is a subset of the upper one. Table 2.2 summarizes our findings on the number of subsets so far.

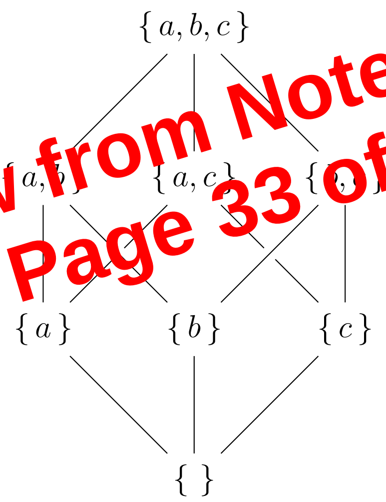


Figure 2.1: Subsets of $\{ a, b, c \}$.

Exercise 2.12. Guess what the rule is by looking at Table 2.2 and listing all subsets of $\{ a, b, c, d \}$ and $\{ a, b, c, d, e \}$, if necessary.

It seems that if S has n elements, then it has 2^n subsets. This is reinforced by Figure 2.1, where we represented the subsets of a three-element set by the responding to the anagram 'eye' (upper right part). There are two different colourings depending on the e's: we can colour the first 'e' by two colours, and the second 'e' by one colour, therefore there are $2 \cdot 1 = 2$ coloured 'eye's in that group. Similarly, every group contain exactly two coloured anagrams. Thus the number of groups (and the number of uncoloured anagrams) is $\frac{6}{2} = 3$.

Exercise 2.26. How many anagrams does the word 'puppy' have? Try to use the argument presented above.

This argument can now be generalized when more letters can be the same:

Theorem 2.8. *Let us assume that a word consists of k different letters, such that there are n_1 of the first letter, n_2 of the second letter, etc. Let $n = n_1 + n_2 + \cdots + n_k$ be the number of letters altogether in this word. Then the number of anagrams this word has is exactly*

$$\frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}.$$

Proof. Let us color all the letters with different colours, and let us count first the number of coloured anagrams. This is the number of permutations of n different letters, that is, $n!$ by Theorem 2.7.

Now, group together those anagrams which represent the same word, and differ only in their colourings. The number of uncoloured anagrams is the same as the number of groups. To compute this number, we count the number of coloured words in each group.

Take an arbitrary group representing an anagram. The words listed in this group differ only by the colourings. The first letter appears n_1-many times, and these letters have $n_1!$-many different colourings by Theorem 2.7. Similarly, the second letter appears n_2-many times, and these letters have $n_2!$-many different colourings by Theorem 2.7, etc. Finally, the kth letter appears n_k-many times, and these letters have $n_k!$-many different colourings by Theorem 2.7. Thus, the number of words in a group is $n_1! \cdot n_2! \cdot \cdots \cdot n_k!$. Therefore the number of groups, and hence the number of (uncoloured) anagrams is

$$\frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}.$$

2.7 Distributing money

Three pirates (Anne Bonney, Black Bellamy and Calico Jack) raid a small ship. They take all the treasure they can find, which is seven gold pieces altogether. Afterwards, they would like to distribute the loot among themselves. They only have one rule: since everybody was useful during the raid, everyone should receive at least 1 gold piece. How many ways can they distribute the seven gold pieces? Gold pieces are identical, it does not matter who gets which gold piece. It only matters how many gold pieces each pirate gets.

One way to solve this problem is of course to write down all possible distributions. Let us list the possibilities by considering the amount of gold pieces received by the highest rewarded pirate. If everyone needs to get at least one gold piece, then nobody can have more than five gold pieces. In fact, if somebody gets five gold pieces, then the other two will have two gold pieces to distribute, which they can only do by giving one gold piece to each of them. This is three possibilities (depending on who receives the five gold pieces). If the pirate in the highest regard gets four gold pieces, then the other two pirates will have three gold pieces to distribute. They can only distribute it as two-one. This altogether amounts to 6 possibilities: 3 possibilities on who gets four gold pieces, then in each case 2 possibilities on who gets two gold pieces, that is, $3 \cdot 2$ possibilities. (Note that this is the number of permutations of the three pirates.) Finally, if the highest reward is three gold pieces, then the other two pirates can distribute the remaining four gold pieces in two different ways: either one of them gets three gold pieces, and the other gets one, or both get two gold pieces. Both distributions amount to 3 possibilities altogether. In the first case there are 3 possibilities to choose who gets one gold piece (and the other two gets three gold pieces each). In the second case there are 3 possibilities to choose who gets three gold pieces (and the other two gets two gold pieces each). Table 2.4 summarizes the 15 possible distributions.

Here, we received exactly the same number of distributions as for the earlier case, when the three pirates needed to distribute 7 gold pieces, and everybody needed to get at least one. This can hardly be a coincidence. Somehow, we should be able to reduce the new problem to the earlier problem. The main difference is that now every pirate needs to get at least two gold pieces instead of one. This can be easily remedied: everyone takes one gold piece at the very beginning. Then seven gold pieces remain $(10 - 3)$, and everyone needs to get at least one more. And this is now exactly the same problem as before. Again, the argument works in general: if there are n pirates and k gold pieces, and everybody needs to get at least two gold pieces, then first every pirate takes one gold piece. This way, everyone needs to get one more gold piece, and they will have $k - n$ gold pieces to distribute further. Applying Theorem 2.15 we can prove

Proposition 2.16. *Assume n pirates want to distribute k gold pieces among themselves (for some $k \geq 2n$) such that everybody gets at least two gold pieces. They can do this in $\binom{k-n-1}{n-1}$-many ways.*

Exercise 2.40. Prove Proposition 2.16 precisely.

The three pirates continued to raid ships. Next time they found a small boat with a fisherman and only four gold pieces. They, again, want to distribute these gold pieces among themselves. But this time they do not want to impose any conditions on the distributions. It may be possible that somebody does not receive any gold pieces, even that somebody takes all the gold. How many ways can they distribute the four gold pieces among themselves?

After the previous two exercises, it is not too difficult to find all the possibilities. There are three possibilities corresponding to the distribution where one of them gets all four gold pieces (three possibilities depending on who gets all the gold). If one of them gets three gold pieces, then the remaining one gold piece goes to one of the remaining pirates. There are 6 such possibilities: 3 choices on who gets three gold pieces, and for each choice there are 2 choices on who of the remaining two pirates gets 1 gold piece (and the last pirate does not get any gold pieces). If the highest rewarded pirate gets two gold pieces, then the remaining two gold pieces can be distributed

Chapter 3

Proof techniques

3.1 Proofs by induction

In mathematics one often uses induction to prove general statements. Let us see how this argument works. Suppose we have a statement $S(n)$ which depends on n. When we apply induction we prove that $S(n_0)$ is true for the smallest possible value n_0. Then we show that if the statement is true for all possible values less than n, then the statement is also true for n. Finally, we conclude that the statement is true for all $n \geq n_0$. There is a very similar notion called recursion. For example we can define $n!$ as follows

$$n! = \begin{cases} 1 & \text{if } n = 1, \\ n \cdot (n-1)! & \text{if } n > 1. \end{cases}$$

The basic idea is that we can compute e.g. 100! if we have computed 99!, 98!, ..., 1!. Induction works in the same way, if we can prove a statement for certain smaller instances, then we can prove it for large values as well. More about recursion will follow in Chapter 5.

Now we study induction in more detail.

Theorem 3.1 (Mathematical Induction I). *Let $S(n)$ be a statement depending on $n \in \mathbb{N}$. Suppose that*

(a) $S(1)$ is true,

3.2 Proofs by contradiction

In this section we study an important tool to prove mathematical theorems. This tool is called proof by contradiction or indirect proof. There is a simple logic behind, instead of proving that something must be true, we prove it indirectly by showing that it cannot be false. We assume that the opposite of our theorem is true. From this assumption we try to obtain such a conclusion which is known to be false. This contradiction then shows that our theorem must be true.

Let us consider a basic example. We try to prove that $\sqrt{2}$ is irrational. We provide an indirect proof. We assume the opposite of our statement, that is, that $\sqrt{2}$ is rational. Rational numbers can be written as $\frac{a}{b}$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that the greatest common divisor of a and b is 1. So we have

$$\sqrt{2} = \frac{a}{b}.$$

Hence $a^2 = 2b^2$. It follows that 2 divides a, so $a = 2a_1$ for some $a_1 \in \mathbb{Z}$. We substitute this into the equation $a^2 = 2b^2$ and we get $4a_1^2 = 2b^2$. After dividing by 2 we get $2a_1^2 = b^2$. So we have that 2 divides b. We have a contradiction since the greatest common divisor of a and b should be 1, but we obtained that 2 divides a and also divides b. Hence 2 divides the greatest common divisor. This contradiction shows that our statement must be true, that is, $\sqrt{2}$ is irrational.

In Section 1.3 there is a statement about the Division algorithm which says that given two integers a and b such that $b > 0$, there exist unique integers q and r for which

$$a = qb + r, \quad 0 \leq r < b.$$

Now we prove that q and r are unique. We give a proof by contradiction. Assume that there exist integers q, q' and r, r' such that $q \neq q'$ or $r \neq r'$ and

$$a = qb + r, \quad 0 \leq r < b,$$
$$a = q'b + r', \quad 0 \leq r' < b.$$

Exercise 3.14. Prove that if $x + y > 10$ for some $x, y \in \mathbb{Z}$, then $x > 5$ or $y > 5$.

Exercise 3.15. Prove that there exists no integer n such that $n^2 - 2$ is a multiple of 4.

Exercise 3.16. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Exercise 3.17. Prove that if a, b and c are odd integers, then the equation

$$ax^2 + bx + c = 0$$

has no solution with $x \in \mathbb{Q}$.

Exercise 3.18. Given n integers $a_1, a_2, \ldots, a_n$, prove that there exists $1 \leq i \leq n$ such that

$$a_i \geq \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

Exercise 3.19. Let F_n be a sequence defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}, n \geq 3$, that is, the Fibonacci sequence. Prove that $\gcd(F_n, F_{n+1}) = 1$ for all positive integer n.

3.3 Constructive proofs

In this section we deal with several problems for which a method can be provided to create a solution. We consider the coin problem (known also as the Frobenius problem). Let us be given a currency system with $k \geq 2$ distinct integer denominations $a_1 < a_2 < \ldots < a_k$. Which amounts can be changed? This question yields the following linear Diophantine equation

$$a_1 x_1 + a_2 x_2 + \ldots + a_k x_k = n,$$

where $x_1, \ldots, x_k$ are non-negative integers. Now we study the case $k = 2$, that is, our equation is

$$a_1 x_1 + a_2 x_2 = n.$$

There are some natural questions to pose:

We would like to have non-negative solutions, hence

$$-3n - 11t \geq 0 \Rightarrow t \leq \frac{-3n}{11}$$
$$2n + 7t \geq 0 \Rightarrow t \geq \frac{-2n}{7}.$$

So we have the following inequalities

$$\frac{-2n}{7} \leq t \leq \frac{-3n}{11}.$$

If there is an integer contained in the interval $[\frac{-2n}{7}, \frac{-3n}{11}]$, then n can be represented in the form $7x_1+11x_2$. Denote by I_n the set $\{\, t \mid \frac{-2n}{7} \leq t \leq \frac{-3n}{11}, t \in \mathbb{Z}\,\}$.

n	I_n	n	I_n	n	I_n	n	I_n	n	I_n
1	$\emptyset$	16	$\emptyset$	31	$\emptyset$	46	$\{-13\}$	61	$\{-17\}$
2	$\emptyset$	17	$\emptyset$	32	$\{-9\}$	47	$\{-13\}$	62	$\{-17\}$
3	$\emptyset$	18	$\{-5\}$	33	$\{-9\}$	48	$\emptyset$	63	$\{-18\}$
4	$\emptyset$	19	$\emptyset$	34	$\emptyset$	49	$\{-14\}$	64	$\{-18\}$
5	$\emptyset$	20	$\emptyset$	35	$\{-10\}$	50	$\{-14\}$	65	$\{-18\}$
6	$\emptyset$	21	$\{-6\}$	36	$\{-10\}$	51	$\{-14\}$	66	$\{-18\}$
7	$\{-2\}$	22	$\{-6\}$	37	$\emptyset$	52	$\emptyset$	67	$\{-19\}$
8	$\emptyset$	23	$\emptyset$	38	$\emptyset$	53	$\{-15\}$	68	$\{-19\}$
9	$\emptyset$	24	$\emptyset$	39	$\{-11\}$	54	$\{-15\}$	69	$\{-19\}$
10	$\emptyset$	25	$\{-7\}$	40	$\{-11\}$	55	$\{-15\}$	70	$\{-20\}$
11	$\{-3\}$	26	$\emptyset$	41	$\emptyset$	56	$\{-16\}$	71	$\{-20\}$
12	$\emptyset$	27	$\emptyset$	42	$\{-12\}$	57	$\{-16\}$	72	$\{-20\}$
13	$\emptyset$	28	$\{-8\}$	43	$\{-12\}$	58	$\{-16\}$	73	$\{-20\}$
14	$\{-4\}$	29	$\{-8\}$	44	$\{-12\}$	59	$\emptyset$	74	$\{-21\}$
15	$\emptyset$	30	$\emptyset$	45	$\emptyset$	60	$\{-17\}$	75	$\{-21\}$

We can find 7 consecutive integers indicated in the table for which the set I_n is not empty, that is, those integers can be represented in the form $7x_1+11x_2$:

$$n = 60 \qquad x_1 = (-3)\cdot 60 - 11\cdot(-17) = 7, x_2 = 2\cdot 60 + 7\cdot(-17) = 1,$$
$$n = 61 \qquad x_1 = (-3)\cdot 61 - 11\cdot(-17) = 4, x_2 = 2\cdot 61 + 7\cdot(-17) = 3,$$
$$n = 62 \qquad x_1 = (-3)\cdot 62 - 11\cdot(-17) = 1, x_2 = 2\cdot 62 + 7\cdot(-17) = 5,$$

for some $t \in \mathbb{Z}$. It remains to determine the integer solutions of the equation $y_1 = 4x_1 + 5x_2 = n + 7t$. The first thing to do is to find a particular solution. It is easy to check that

$$x_1 = -n + 3t,$$
$$x_2 = n - t$$

is a solution. Applying the techniques used in case of two variables we get the following parametrization of integral solution

$$x_1 = -n + 3t - 5s,$$
$$x_2 = n - t + 4s,$$
$$x_3 = -t$$

for some $s, t \in \mathbb{Z}$. As a concrete example consider the equation $4x_1 + 5x_2 + 7x_3 = 23$. Then we obtain integer solutions by substituting concrete integral values into the above formulas. Some solutions are indicated in the following table

(s, t)	(x_1, x_2, x_3)
$(0, 0)$	$(-23, 23, 0)$
$(-1, 0)$	$(-18, 19, 0)$
$(0, -1)$	$(-26, 24, 1)$
$(1, 0)$	$(-28, 27, 0)$
$(0, 1)$	$(-20, 22, -1)$
$(-1, -1)$	$(-21, 20, 1)$
$(1, 1)$	$(-25, 26, -1)$

What about non-negative integer solutions? That is, if one asks for solutions such that $x_1, x_2, x_3 \in \mathbb{N} \cup \{0\}$. In case of the equation $4x_1 + 5x_2 + 7x_3 = n$ we determined the parametrization of the integral solutions, so we get the following inequalities

$$0 \le -n + 3t - 5s,$$
$$0 \le n - t + 4s,$$
$$0 \le -t.$$

Proposition 3.9. *There is a nonzero multiple of 6 whose digits are all zeroes and ones.*

Proof. We apply the pigeonhole principle and the Division algorithm. Consider the integers $a_n = \sum_{k=0}^{n} 10^k$ for $n = 0, 1, 2, 3, 4, 5$. We can write these numbers as $q_n \cdot 6 + r_n$, where q_n is the quotient and r_n is the remainder, so $0 \leq r_n < 6$. There are six possibilities for r_n and there are six integers $a_0, a_1, \ldots, a_5$. The numbers $a_0, a_1, \ldots, a_5$ are odd integers while 6 is even, hence $r_n \neq 0$ for all n. We have that $r_n \in \{1, 2, 3, 4, 5\}$ for all n. There are only 5 pigeonholes (possible remainders) and 6 pigeons (integers a_n). We obtain that there are at least two integers having the same remainder, say, a_{m_1} and a_{m_2}, where $m_1 < m_2$. In this case $a_{m_2} - a_{m_1}$ is divisible by 6 and all the digits are zeroes and ones.

n	a_n	$q_n \cdot 6 + r_n$
0	1	$0 \cdot 6 + 1$
1	11	$1 \cdot 6 + 5$
2	111	$18 \cdot 6 + 3$
3	1111	$185 \cdot 6 + 1$
4	11111	$1851 \cdot 6 + 5$
5	111111	$18518 \cdot 6 + 3$

It is clear that $r_0 = r_3 = 1$, therefore $a_3 - a_0 = 1111 - 1 = 1110$ is a multiple of 6 ($1110 = 185 \cdot 6$) and this integer is in a right form. □

Proposition 3.10. *Let A be a set containing $n \geq 2$ integers. There is a subset of A such that the sum of its elements is a multiple of n.*

Proof. We have a set containing n elements, let us say these are $a_1, a_2, \ldots, a_n$. We define n subsets as follows

$$S_k = \{ a_1, \ldots, a_k \}, \quad k = 1, 2, \ldots, n,$$

that is, $S_1 = \{ a_1 \}, S_2 = \{ a_1, a_2 \}, \ldots, S_n = A$. Denote by s_k the sum of the elements of S_k. We apply the Division algorithm to write $s_k = q_k \cdot n + r_k$, where $0 \leq r_k < n$. If for some k we have $r_k = 0$, then

$$s_k = a_1 + \ldots + a_k = q_k \cdot n,$$

(every row starts with the zeroth number), the two numbers above it are the $(k-1)$st and kth of row $n-1$, that is, $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$. Thus, if we prove that $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$, then the two triangles are indeed the same.

Proposition 4.1. *For positive integers $k \le n$ we have*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. Let us substitute the formula (2.1) into the right-hand side:

$$\begin{aligned}
&\binom{n-1}{k-1} + \binom{n-1}{k} \\
&= \frac{(n-1)!}{(k-1)! \cdot (n-1-(k-1))!} + \frac{(n-1)!}{k! \cdot (n-1-k)!} \\
&= \frac{(n-1)!}{(k-1)! \cdot (n-k)!} + \frac{(n-1)!}{k! \cdot (n-k-1)!} \\
&= \frac{(n-1)! \cdot k + (n-1)! \cdot (n-k)}{k! \cdot (n-k)!} = \frac{(n-1)! \cdot (k+n-k)}{k! \cdot (n-k)!} \\
&= \frac{(n-1)! \cdot n}{k! \cdot (n-k)!} = \frac{n!}{k! \cdot (n-k)!} = \binom{n}{k}.
\end{aligned}$$

□

Exercise 4.1. Create a precise proof using induction that the two triangles are the same.

This proof is a correct one, but not necessarily satisfying. It contains calculations, but does not show the reason *why* the sum of the binomial coefficients $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$ is really $\binom{n}{k}$. One might wonder if there is an "easier" proof, which only uses the definition of $\binom{n}{k}$. Indeed there is, as we show now.

Second proof of Proposition 4.1. Let $A = \{1, 2, \ldots, n\}$, and we count the number of k-element subsets of A in two different ways. On the one hand, we know that the number of k-element subsets of A is $\binom{n}{k}$. On the other hand, we count the k-element subsets such that we first count those which contain the element n, then we count those, which do not.

Let us start by the sum of the numbers in a row:

$$\begin{aligned} 1 &= 1, \\ 1+1 &= 2, \\ 1+2+1 &= 4, \\ 1+3+3+1 &= 8, \\ 1+4+6+4+1 &= 16, \\ 1+5+10+10+5+1 &= 32, \\ 1+6+15+20+15+6+1 &= 64. \end{aligned}$$

It seems from these equations that the sum of the numbers in the nth row is 2^n. This stetement is equivalent to the equality

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-2}+\binom{n}{n-1}+\binom{n}{n}=2^n.$$

Note, that we have already proved this, first in Proposition 2.14, then later in Exercise 4.5. Now, we prove it a third way, using the generating rule of Pascal's triangle.

Let us consider first the 7th row, and try to compute the sum using the generating rule of Pascal's triangle, rather than adding the numbers:

$$\begin{aligned} &1+7+21+35+35+21+7+1 \\ &= 1+(1+6)+(6+15)+(15+20)+(20+15)+(15+6)+(6+1)+1 \\ &= 2\cdot(1+6+15+20+15+6+1) = 2\cdot 2^6 = 2^7 = 128. \end{aligned}$$

This idea can be used in the general case, as well.

Now, we prove that the sum of the numbers in the nth row of Pascal's triangle is 2^n by induction on n. The statement holds for $n=0$ and $n=1$ (in fact, we just calculated that it holds for $n \le 7$). Assume now that the statement holds for n, as well. That is, the sum of the numbers in the nth row is 2^n. Consider the sum of the $(n+1)$st row, and let us use the generating rule of Pascal's triangle:

$$\begin{aligned}
&\binom{n+1}{0}+\binom{n+1}{1}+\binom{n+1}{2}+\cdots+\binom{n+1}{n-1}+\binom{n+1}{n}+\binom{n+1}{n+1}\\
&=\binom{n}{0}+\left(\binom{n}{0}+\binom{n}{1}\right)+\left(\binom{n}{1}+\binom{n}{2}\right)+\left(\binom{n}{2}+\binom{n}{3}\right)+\dots\\
&+\left(\binom{n}{n-2}+\binom{n}{n-1}\right)+\left(\binom{n}{n-1}+\binom{n}{n}\right)+\binom{n}{n}\\
&=2\cdot\left[\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n-2}+\binom{n}{n-1}+\binom{n}{n}\right]\\
&=2\cdot 2^n=2^{n+1}.
\end{aligned}$$

First, we replaced $\binom{n+1}{0}=1$ by $\binom{n}{0}=1$, and $\binom{n+1}{n+1}=1$ by $\binom{n}{n}=1$, then we used the generating rule of Pascal's triangle. Then we observed that every $\binom{n}{k}$ occurs twice in the sum (for $0 \le k \le n$). Finally, we used the induction hypothesis on the sum of the numbers for the nth row.

Let us use a similar reasoning to calculate the sum of the numbers in a row, *with alternating signs*. That is, compute the sum

$$\sum_{k=0}^{n}(-1)^k\cdot\binom{n}{k}=\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^{n-1}\cdot\binom{n}{n-1}+(-1)^n\cdot\binom{n}{n}.$$

It is easy to compute this sum for the first couple rows:

$$\begin{aligned}
1&=1,\\
1-1&=0,\\
1-2+1&=0,\\
1-3+3-1&=0,\\
1-4+6-4+1&=0,\\
1-5+10-10+5-1&=0,\\
1-6+15-20+15-6+1&=0,\\
1-7+21-35+35-21+7-1&=0.
\end{aligned}$$

It seems likely that for $n \ge 1$ the alternating sum of the numbers in the nth row of Pascal's triangle is 0.

Exercise 4.11. Prove that

$$\sum_{k=0}^{l}\binom{n}{k}\cdot\binom{m}{l-k}=\binom{n+m}{l},\ \text{that is,}$$

$$\binom{n}{0}\cdot\binom{m}{l}+\binom{n}{1}\cdot\binom{m}{l-1}+\cdots+\binom{n}{l}\cdot\binom{m}{0}=\binom{n+m}{l}. \tag{4.6}$$

How do we need to choose m and l so that (4.6) gives us the equality (4.5)?

We could have used the Binomial theorem to prove (4.5):

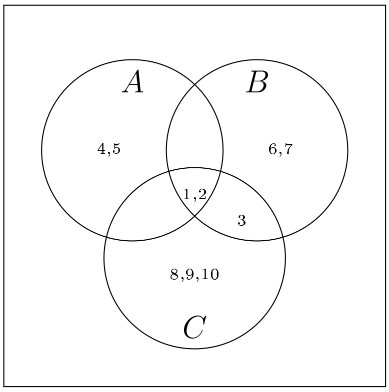
Second proof of Proposition 4.4. Consider $(x+y)^{2n}$, and expand it using the Binomial theorem:

$$(x+y)^{2n}=\sum_{k=0}^{2n}\binom{2n}{k}x^{2n-k}\cdot y^k.$$

Then the right hand side of (4.5) is the coefficient of the term $x^n y^n$. We prove that the left hand side is the coefficient of $x^n y^n$ as well. For this, we compute $(x+y)^{2n}$ by multiplying $(x+y)^n \cdot (x+y)^n$ after expanding both factors using the Binomial theorem:

$$(x+y)^{2n}=(x+y)^n\cdot(x+y)^n=\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right)\cdot\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right).$$

Now, let us compute the coefficient of $x^n y^n$. When do we obtain $x^n y^n$ when we multiply $\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right)$ by itself? Take for example x^n from the first factor, this must be multiplied by y^n from the second factor to obtain $x^n y^n$. The coefficient of x^n in the first factor is $\binom{n}{0}$, the coefficient of y^n in the second factor is $\binom{n}{n}$, thus this multiplication contributes by $\binom{n}{0}\cdot\binom{n}{n}$ to the coefficient of $x^n y^n$ in $(x+y)^{2n}$. Similarly, take the term $x^{n-1}y$ from the first factor, this must be multiplied by xy^{n-1} from the second factor to obtain $x^n y^n$. The coefficient of $x^{n-1}y$ in the first factor is $\binom{n}{1}$, the coefficient of xy^{n-1} in the second factor is $\binom{n}{n-1}$, thus this multiplication contributes by $\binom{n}{1}\cdot\binom{n}{n-1}$ to the coefficient of $x^n y^n$ in $(x+y)^{2n}$. In general, for some k the term $x^{n-k}y^k$ in the first factor must be multiplied by $x^k y^{n-k}$ from the second factor. The coefficient of $x^{n-k}y^k$ in the first factor is $\binom{n}{k}$, the coefficient of $x^k y^{n-k}$ in the second factor is $\binom{n}{n-k}$, thus this multiplication contributes by



1.9 (a) $\sum_{i=4}^{7} i = 4 + 5 + 6 + 7$,

(b) $\sum_{i=1}^{5} (i^2 - i) = 0 + 2 + 6 + 12 + 20$,

(c) $\sum_{i=1}^{4} 10^i = 10 + 100 + 1000 + 10000$,

(d) $\sum_{2 \le i \le 5} \frac{1}{2^i} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$,

(e) $\sum_{i \in S} (-1)^i$, where $S = \{2, 3, 5, 8\}$ is $1 + (-1) + (-1) + 1$.

1.10 (a) $2 + 4 + 6 + 8 + 10 = \sum_{i=1}^{5} 2i$,

(b) $1 + 4 + 7 + 10 = \sum_{i=0}^{3} (3i + 1)$,

(c) $\frac{1}{4} + \frac{1}{2} + 1 + 2 + 4 = \sum_{i=-2}^{2} 2^i$,

(d) $\frac{1}{4} - \frac{1}{2} + 1 - 2 + 4 = \sum_{i=-2}^{2} (-2)^i$.

1.11 (a) $\prod_{i=-4}^{-1} i = (-4) \cdot (-3) \cdot (-2) \cdot (-1)$,

(b) $\prod_{i=1}^{4} (i^2) = 1 \cdot 4 \cdot 9 \cdot 16$,

(c) $\prod_{i=1}^{3} 2^i = 2 \cdot 4 \cdot 8$,

(d) $\prod_{-2 \le i \le 3} \frac{1}{2^i} = 4 \cdot 2 \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{8}$,

(e) $\prod_{i \in S} (-1)^i$, where $S = \{2, 4, 6, 7\}$ is $(-1)^2 \cdot (-1)^4 \cdot (-1)^6 \cdot (-1)^7$.

1.12 (a) $1 \cdot 3 \cdot 5 \cdot 7 = \prod_{i=0}^{3} (2i + 1)$,

(b) $(-1) \cdot 2 \cdot 5 \cdot 8 = \prod_{i=0}^{3} (3i - 1)$,

(c) $\frac{1}{9} \cdot \frac{1}{3} \cdot 1 \cdot 3 \cdot 9 = \prod_{i=-2}^{2} 3^i$.

If $n \geq 2$, then another proof could be

$$n! = n \cdot \underbrace{(n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1}_{(n-1)!} = n \cdot (n-1)!.$$

Nevertheless, the claim is true for $n = 1$, as well:

$$1! = 1 = 1 \cdot 1 = 1 \cdot 0!.$$

1.16 (a) We obtain that

$$\begin{aligned} 678 &= 1 \cdot 567 + 111 \\ 567 &= 5 \cdot 111 + 12 \\ 111 &= 9 \cdot 12 + 3 \\ 12 &= 4 \cdot 3 + 0. \end{aligned}$$

Thus $\gcd(678, 567) = 3$. We work backwards to compute x and y :

$$\begin{aligned} 3 &= 111 - 9 \cdot 12 \\ &= 111 - 9 \cdot (567 - 5 \cdot 111) = -9 \cdot 567 + 46 \cdot 111 \\ &= -9 \cdot 567 + 46 \cdot (678 - 567) = 46 \cdot 678 - 55 \cdot 567. \end{aligned}$$

Hence we have

$$46 \cdot 678 - 55 \cdot 567 = \gcd(678, 567) = 3.$$

(b) We get that

$$\begin{aligned} 803 &= 2 \cdot 319 + 165 \\ 319 &= 1 \cdot 165 + 154 \\ 165 &= 1 \cdot 154 + 11 \\ 154 &= 14 \cdot 11 + 0. \end{aligned}$$

It follows that $\gcd(803, 319) = 11$. Now we find x and y :

$$\begin{aligned} 11 &= 165 - 154 \\ &= 165 - (319 - 165) = -319 + 2 \cdot 165 \\ &= -319 + 2 \cdot (803 - 2 \cdot 319) = 2 \cdot 803 - 5 \cdot 319. \end{aligned}$$

to divide it by 2. But 15/2 is not an integer, while the number of handshakes should be an integer. This contradiction proves that it is not possible that each of 5 people shakes hand with 3 others.

For 7 people we can use this argument, again. If we sum up all the handshakes for everyone, we obtain $7 \cdot 3 = 21$, as each of the 7 people shakes hand with 3 others. This way, we counted every handshake twice, thus to obtain the number of handshakes we need to divide it by 2. But 21/2 is not an integer, while the number of handshakes should be an integer. This contradiction proves that it is not possible that each of 7 people shakes hand with 3 others.

2.3 The four boys shake hands with each other, that is, $\frac{4 \cdot 3}{2} = 6$ handshakes. The four girls kisses each other, those are $\frac{4 \cdot 3}{2} = 6$ kisses by the same formula we use for handshakes. Finally, a boy and a girl kisses, as well. All four boys kiss all four girls on the cheek, which is $4 \cdot 4 = 16$ more kisses. Ultimately, there are 6 handshakes and 22 kisses.

2.4 (a) Not possible. If there are five packs, each of them containing odd many rabbits, then altogether in the five packs there are odd many rabbits (odd+odd+odd+odd+odd is odd). As 100 is not an odd number, it is not possible to do the required distribution.

(b) It is possible, e.g. $3 \cdot 3 \cdot 1 \cdot 1 \cdot 1$. Another possibility could be $9 \cdot 1 \cdot (-1) \cdot 1 \cdot (-1)$, or simply 9 (as only one integer).

(c) It is possible, e.g. $3 \cdot 3 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot (-1) \cdot 1 \cdot (-1)$, or another possibility is $9 \cdot 1 \cdot (-1) \cdot 1 \cdot (-1) \cdot 1 \cdot (-1) \cdot 1 \cdot (-1)$.

(d) Not possible. If the product of integer numbers is 9, then all of them are odd. But then the sum of 9 odd integer numbers is odd again, and hence cannot be 0.

2.5 (a) We can apply Proposition 2.1 and obtain

$$1 + 2 + 3 + \cdots + 23 + 24 = \frac{24 \cdot 25}{2} = 300.$$

2.17 After computing the binary representation, we just add the elements corresponding to the places where the digits are 1.

decimal number	binary number	subset of S
11	01011_2	$\{ a_0, a_1, a_3 \}$
7	00111_2	$\{ a_0, a_1, a_2 \}$
15	01111_2	$\{ a_0, a_1, a_2, a_3 \}$
16	10000_2	$\{ a_4 \}$
31	11111_2	$\{ a_0, a_1, a_2, a_3, a_4 \}$

Note, that the encoding was defined in such a way, that the subset of $\{ a_0, a_1, a_2, a_3 \}$ corresponding to k is the same as the subset of $\{ a_0, a_1, a_2, a_3, a_4 \}$ corresponding to k (for arbitrary $0 \le k \le 15$).

2.18 After computing the binary representation, we just add the elements corresponding to the places where the digits are 1.

decimal number	binary number	subset of S
49	110001_2	$\{ a_0, a_4, a_5 \}$

2.19 After computing the binary representation, we just add the elements corresponding to the places where the digits are 1.

decimal number	binary number	subset of S
101	1100101_2	$\{ a_0, a_2, a_5, a_6 \}$

2.20 After computing the binary representation, we just add the elements corresponding to the places where the digits are 1.

decimal number	binary number	subset of S
199	11000111_2	$\{ a_0, a_1, a_2, a_6, a_7 \}$

2.21 All possibilities are listed in Table 6.2 on page 149.

2.22 The number of permutations of $\{ 1, 2, 3, 4 \}$ is $4! = 24$.

2.23 The number of permutations of $\{ a, b, c, d \}$ is $4! = 24$.

The boys can sit on their seats in $5! = 120$-many ways. The girls (independently on how the boys sit) can sit on their seats in $3! = 6$-many ways. Altogether, they can sit in $6 \cdot 120 = 720$-many ways.

2.25 The number of anagrams of 'retinas' is the same as the number of permutations of the letters 'r', 'e', 't', 'i', 'n', 'a' and 's'. There are 7 different letters, hence the number of permutations is $7! = 5\ 040$.

2.26 Again, let us color the 'p's in the anagrams by three colors: red, green, blue. This way, there will be $5! = 120$-many coloured anagrams of puppy, the same as the number of permutations of five different elements. Now, group together those anagrams, which only differ by their colouring. For example the group 'puppy' would contain 'puppy', 'puppy', 'puppy', 'puppy', 'puppy', 'puppy'. How do we know that there are six coloured 'puppy's? The coloured 'puppy's only differ in the colourings of the 'p's. The first 'p' can be coloured by 3 different colours, the next 'p' (right after the 'u') can be coloured by two different colours (it cannot be coloured by the same colour as the first 'p'), then the last 'p' should be coloured by the remaining colour. Thus, there are $3 \cdot 2 \cdot 1 = 6$-many coloured 'puppy's. Similarly, there are 6 coloured versions of every anagram. Therefore there are $\frac{120}{6} = 20$ (uncoloured) anagrams of 'puppy'. These are 'pppuy', 'pppyu', 'ppupy', 'ppypu', 'ppuyp', 'ppyup', 'puppy', 'pyppu', 'pupyp', 'pypup', 'puypp', 'pyupp', 'upppy', 'ypppu', 'uppyp', 'yppup', 'upypp', 'ypupp', 'uyppp', 'yuppp'.

2.27 (a) The word 'college' contains 7 letters, two of them are 'e's and two of them are 'l's, thus the number of anagrams is

$$\frac{7!}{2! \cdot 2!} = \frac{5\ 040}{2 \cdot 2} = 1\ 260.$$

(b) The word 'discrete' contains 8 letters, two of them are 'e's, thus the number of anagrams is

$$\frac{8!}{2!} = \frac{40\ 320}{2} = 20\ 160.$$

2.33 The two numbers are equal, as the following calculation shows

$$\frac{90!}{5!\cdot 85!}=\frac{90\cdot 89\cdot 88\cdot 87\cdot 86\cdot 85!}{5!\cdot 85!}=\frac{90\cdot 89\cdot 88\cdot 87\cdot 86}{5!}.$$

2.34 The required binomial coefficients are computed and arranged into a triangle in Table 6.3 on page 201.

2.35 By the definition

$$\binom{n}{0}=\frac{n!}{0!\cdot(n-0)!}=\frac{n!}{0!\cdot n!}=\frac{1}{0!}=\frac{1}{1}=1,$$

$$\binom{n}{1}=\frac{n!}{1!\cdot(n-1)!}=\frac{n\cdot(n-1)!}{1!\cdot(n-1)!}=\frac{n}{1!}=\frac{n}{1}=n,$$

$$\binom{n}{2}=\frac{n!}{2!\cdot(n-2)!}=\frac{n\cdot(n-1)\cdot(n-2)!}{2!\cdot(n-2)!}=\frac{n\cdot(n-1)}{2!}=\frac{n\cdot(n-1)}{2},$$

$$\binom{n}{n-2}=\frac{n!}{(n-2)!\cdot(n-(n-2))!}=\frac{n\cdot(n-1)\cdot(n-2)!}{(n-2)!\cdot 2!}=\frac{n\cdot(n-1)}{2!}$$
$$=\frac{n\cdot(n-1)}{2},$$

$$\binom{n}{n-1}=\frac{n!}{(n-1)!\cdot(n-(n-1))!}=\frac{n\cdot(n-1)!}{(n-1)!\cdot 1!}=\frac{n}{1!}=\frac{n}{1}=n,$$

$$\binom{n}{n}=\frac{n!}{n!\cdot(n-n)!}=\frac{n!}{n!\cdot 0!}=\frac{1}{0!}=\frac{1}{1}=1.$$

2.36 Using Table 6.3 from Exercise 2.34, it is not hard to determine the required sums:

$$\sum_{k=0}^{0}\binom{0}{k}=\binom{0}{0}=1,$$

$$\sum_{k=0}^{1}\binom{1}{k}=\binom{1}{0}+\binom{1}{1}=1+1=2,$$

$$\sum_{k=0}^{2}\binom{2}{k}=\binom{2}{0}+\binom{2}{1}+\binom{2}{2}=1+2+1=4,$$

$$\sum_{k=0}^{3}\binom{3}{k}=\binom{3}{0}+\binom{3}{1}+\binom{3}{2}+\binom{3}{3}=1+3+3+1=8,$$

$$\sum_{k=0}^{4}\binom{4}{k}=\binom{4}{0}+\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4}$$

(g)

$$100^{10} = 10^{20},$$

(h)

$$10^{100}.$$

On the other hand, by the induction hypothesis

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \left(\sum_{i=1}^{k} \frac{1}{i(i+1)} \right) + \frac{1}{(k+1)(k+2)} =$$
$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} =$$
$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

Therefore $S(k+1)$ is true and the identity

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

is valid for all positive integers n.

3.8 Let us compute the first few elements of the sequence

n	a_n
1	1
2	8
3	$a_2 + 2a_1 = 10$
4	$a_3 + 2a_2 = 26$
5	$a_4 + 2a_3 = 46$

Now we compute the values of the formula $\frac{3}{2} \cdot 2^n + 2 \cdot (-1)^n$ for $n \in \{1, 2, 3, 4, 5\}$

n	$\frac{3}{2}2^n + 2(-1)^n$
1	1
2	8
3	10
4	26
5	46

We checked that $a_n = \frac{3}{2} \cdot 2^n + 2 \cdot (-1)^n$ for $n \in \{1, 2, 3, 4, 5\}$. Assume that the statement is true for $S(k-1)$ and $S(k)$ for some $2 \leq k \in \mathbb{N}$, By the induction hypothesis $a_k \leq 2$, hence

$$a_{k+1} \leq \sqrt{2+2} = 2.$$

The statement $S(k+1)$ has been proved and thus we have that $a_n \leq 2$ for all $n \in \mathbb{N}$.

3.11 We prove the statement by induction. If $n = 1$, then the 1-digit integer $a_1 = 2$ is divisible by 2. Therefore the statement is true for $n = 1$. It is not difficult to deal with the case $n = 2$. There are only four possible integers

$$a_1a_2 \in \{\, 11, 12, 21, 22 \,\}.$$

It is easy to see that 2^2 divides 12. Assume that the statement is true for some $1 \leq k \in \mathbb{N}$, that is, there exists a k-digit integer $a_1a_2\ldots a_k$ which is a multiple of 2^k. Let us consider the statement for $k+1$. By induction hypothesis we have

$$a_1a_2\ldots a_k = 2^k \cdot A.$$

We claim that either

$$10^k + a_1a_2\ldots a_k = 1a_1a_2\ldots a_k$$

or

$$2 \cdot 10^k + a_1a_2\ldots a_k = 2a_1a_2\ldots a_k$$

is a multiple of 2^{k+1}. We can rewrite the above integers as follows

$$10^k + a_1a_2\ldots a_k = 10^k + 2^k \cdot A = 2^k(5^k + A),$$
$$2 \cdot 10^k + a_1a_2\ldots a_k = 2 \cdot 10^k + 2^k \cdot A = 2^k(2 \cdot 5^k + A).$$

If A is odd, then $5^k + A$ is even. In this case

$$1a_1a_2\ldots a_k$$

is an integer having $k+1$ digits and it is divisible by 2^{k+1}. If A is even, then $2 \cdot 5^k + A$ is even. That is,

$$2a_1a_2\ldots a_k$$

is a $(k+1)$-digit number which is a multiple of 2^{k+1}.

If $n = 1$, then the left-hand side is $F_2 = 1$ and the right-hand side is $F_3 - 1 = 2 - 1 = 1$. So the identity is valid. Assume that for some $1 \leq k \in \mathbb{N}$ the identity holds, that is,

$$F_2 + F_4 + \ldots + F_{2k} = F_{2k+1} - 1.$$

Let us handle the sum for $k+1$ terms, that is, the sum

$$F_2 + F_4 + \ldots + F_{2k} + F_{2k+2}.$$

It can be written as

$$(F_2 + F_4 + \ldots + F_{2k}) + F_{2k+2} = F_{2k+1} - 1 + F_{2k+2} = F_{2k+3} - 1.$$

Thus the identity has been proved for all positive integers.

3.13 (a) First compute F_{3n} for some n, let say for $n = 1, 2, 3$. We have

$$F_3 = 2,$$
$$F_6 = 8,$$
$$F_9 = 34.$$

We checked that F_{3n} is even for $n = 1, 2, 3$. Assume that F_{3k} is even for some $1 \leq k \in \mathbb{N}$. For $k+1$ we have $F_{3(k+1)} = F_{3k+3}$. By definition

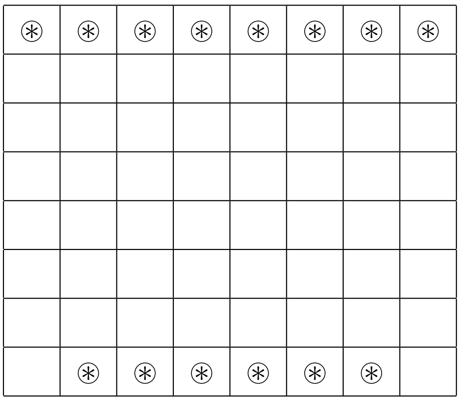
$$F_{3k+3} = F_{3k+2} + F_{3k+1} = F_{3k+1} + F_{3k} + F_{3k+1} = 2 \cdot F_{3k+1} + F_{3k}.$$

By induction F_{3k} is even, so $2 \cdot F_{3k+1} + F_{3k}$ is even. The statement is true.

(b) If $n = 1$, then $F_{5 \cdot 1} = 5$. That is, the property holds for $n = 1$. Assume that F_{5k} is a multiple of 5 for some $1 \leq k \in \mathbb{N}$. For $k+1$ we have

$$F_{5(k+1)} = F_{5k+5} = F_{5k+4} + F_{5k+3} = F_{5k+3} + F_{5k+2} + F_{5k+2} + F_{5k+1} =$$
$$3F_{5k+2} + 2F_{5k+1} = 3(F_{5k+1} + F_{5k}) + 2F_{5k+1} = 5F_{5k+1} + 3F_{5k}.$$

It is clear that 5 divides $5F_{5k+1}$ and induction hypothesis implies that $3F_{5k}$ is a multiple of 5. Therefore 5 divides $5F_{5k+1} + 3F_{5k}$. We proved the property for $k+1$. It follows that F_{5n} is a multiple of 5 for all $n \in \mathbb{N}$.



It remains to show that it is not possible to place more than 14 bishops in such a way that they can not hit each other. A natural idea is to divide the 64 chess squares into 14 groups such that if two bishops are in the same group then they can hit each other. We can produce 14 such groups

4	8	5	9	6	10	7	11
8	4	9	5	10	6	11	7
3	9	4	10	5	11	6	12
9	3	10	4	11	5	12	6
2	10	3	11	4	12	5	13
10	[illegible]	11	3	12	4	13	5
1	11	2	12	3	13	4	14
11	1	12	2	13	3	14	4

3.32 (a) The first card is 7♣, hence the suit of the hidden card is ♣. The distance can be obtained from the following table

distance	order of the 3 cards
1	$3\diamondsuit, J\diamondsuit, A\spadesuit$
2	$3\diamondsuit, A\spadesuit, J\diamondsuit$
3	$J\diamondsuit, 3\diamondsuit, A\spadesuit$
4	$J\diamondsuit, A\spadesuit, 3\diamondsuit$
5	$A\spadesuit, 3\diamondsuit, J\diamondsuit$
6	$A\spadesuit, J\diamondsuit, 3\diamondsuit$

4.7 Using the Binomial theorem we obtain

$$(x+y)^8 = \sum_{k=0}^{8} \binom{8}{k} x^{8-k} y^k = x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3$$
$$+ 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8,$$

$$(x-y)^8 = \sum_{k=0}^{8} \binom{8}{k} x^{8-k} (-y)^k = x^8 - 8x^7y + 28x^6y^2 - 56x^5y^3$$
$$+ 70x^4y^4 - 56x^3y^5 + 28x^2y^6 - 8xy^7 + y^8,$$

$$(a+1)^{10} = \sum_{k=0}^{10} \binom{10}{k} \cdot a^{10-k} \cdot 1^k = a^{10} + 10a^9 + 45a^8 + 120a^7$$
$$+ 210a^6 + 252a^5 + 210a^4 + 120a^3 + 45a^2 + 10a + 1,$$

$$(b-3)^5 = \sum_{k=0}^{5} \binom{5}{k} b^{5-k} (-3)^k = b^5 - 15b^4 + 90b^3$$
$$- 270b^2 + 405b - 243,$$

$$(1+2/x)^5 = \sum_{k=0}^{5} \binom{5}{k} \cdot 1^{5-k} \cdot \left(\frac{2}{x}\right)^k = 1 + \frac{10}{x} + \frac{40}{x^2} + \frac{80}{x^3} + \frac{80}{x^4} + \frac{32}{x^5},$$

$$(a+b)^6 = \sum_{k=0}^{6} \binom{6}{k} a^{6-k} b^k = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3$$
$$+ 15a^2b^4 + 6ab^5 + b^6,$$

$$(1+x)^5 = \sum_{k=0}^{5} \binom{5}{k} \cdot 1^{5-k} \cdot x^k = 1 + 5x + 10x^2$$
$$+ 10x^3 + 5x^4 + x^5,$$

$$(3a+4b)^4 = \sum_{k=0}^{4} \binom{4}{k} \cdot (3a)^{4-k} \cdot (4b)^k = (3a)^4 + 4 \cdot (3a)^3 \cdot (4b)$$
$$+ 6 \cdot (3a)^2 \cdot (4b)^2 + 4 \cdot (3a) \cdot (4b)^3 + (4b)^4 = 81a^4$$
$$+ 432a^3b + 864a^2b^2 + 768ab^3 + 256b^4,$$

$$(3-2x)^4 = \sum_{k=0}^{4} \binom{4}{k} \cdot 3^{4-k} \cdot (-2x)^k = 3^4 - 4 \cdot 3^3 \cdot (2x) + 6 \cdot 3^2 \cdot (2x)^2$$
$$- 4 \cdot 3 \cdot (2x)^3 + (2x)^4 = 81 - 216x + 216x^2 - 96x^3 + 16x^4.$$

For this, we compute $(x+y)^{n+m}$ by multiplying $(x+y)^n \cdot (x+y)^m$ after expanding both factors using the Binomial theorem:

$$(x+y)^{n+m} = (x+y)^n \cdot (x+y)^m = \left(\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\right) \cdot \left(\sum_{k=0}^{m} \binom{m}{k} x^{m-k} y^k\right).$$

Now, let us compute the coefficient of $x^{n+m-l}y^l$. When do we obtain $x^{n+m-l}y^l$ when we multiply $\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k\right)$ by $\left(\sum_{k=0}^{m}\binom{m}{k}x^{m-k}y^k\right)$? For some $0 \le k \le l$ the term $x^{n-k}y^k$ in the first factor must be multiplied by $x^{m-l+k}y^{l-k}$ from the second factor. The coefficient of $x^{n-k}y^k$ in the first factor is $\binom{n}{k}$, the coefficient of $x^{m-l+k}y^{l-k}$ in the second factor is $\binom{m}{l-k}$, thus this multiplication contributes by $\binom{n}{k} \cdot \binom{m}{l-k}$ to the coefficient of $x^{n+m-l}y^l$ in $(x+y)^{n+m}$. That is, the coefficient of $x^{n+m-l}y^l$ in $(x+y)^{n+m}$ is

$$\sum_{k=0}^{l} \binom{n}{k} \cdot \binom{m}{l-k}.$$

Moreover, the coefficient of $x^{n+m-l}y^l$ in $(x+y)^{n+m}$ is $\binom{n+m}{l}$, thus the two numbers must be equal, which proves (4.6):

$$\sum_{k=0}^{n} \binom{n}{k} \binom{m}{l-k} = \binom{n+m}{l}.$$

4.13 We can try to prove the identity by induction on m. For $m = 0$ the identity holds, as the left hand side is $\binom{n}{0} = 1$, the right hand side is $\binom{n+1}{0} = 1$, as well. Assume that the identity holds for $m-1$, that is,

$$\sum_{k=0}^{m-1} \binom{n+k}{k} = \binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+m-1}{m-1} = \binom{n+m}{m-1}.$$

This is the induction hypothesis. Now we prove the identity for m.

$$\sum_{k=0}^{m} \binom{n+k}{k} = \underbrace{\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+m-1}{m-1}}_{=\binom{n+m}{m-1}\text{, by the induction hypothesis}} + \binom{n+m}{m}$$

$$= \binom{n+m}{m-1} + \binom{n+m}{m} = \binom{n+m+1}{m}.$$

5.4 This is an example of an order 3 linear recurrence. We define $g_n = g_0 r^n$ for some g_0 and r, which is a geometric progression. We assume that it satisfies the same recurrence relation as a_n, that is, we obtain

$$r^3 = -2r^2 + r + 2.$$

It is a cubic polynomial. We look for integer solutions. If there is an integral root, then it divides 2. Hence the possible integral roots are $\pm 2, \pm 1$.

r	$r^3 + 2r^2 - r - 2$
-2	0
-1	0
1	0
2	12

The cubic polynomial $r^3 + 2r^2 - r - 2$ can be written as $(x-1) \cdot (x+1) \cdot (x+2)$, that is, there are three integral roots. In this case we have three geometric progressions satisfying the recurrence, therefore the appropriate linear combination is

$$W_n = s \cdot (-2)^n + t \cdot (-1)^n + u \cdot 1^n.$$

The corresponding system of linear equations is

$$s + t + u = 0,$$
$$-2s - t + u = 1,$$
$$4s + t + u = 2.$$

We subtract the first equation from the third one to get $3s = 2$. So we have that $s = 2/3$. We eliminate s from the first two equations

$$t + u = -\frac{2}{3},$$
$$-t + u = \frac{7}{3}.$$

It is easy to see that $u = 5/6$ and $t = -3/2$. The explicit formula for a_n is

$$\frac{2}{3} \cdot (-2)^n - \frac{3}{2} \cdot (-1)^n + \frac{5}{6}.$$