Prove that the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is diagonalizable if $-4bc < (a-d)^2$ and is not diagonalizable if $-4bc > (a-d)^2$

Proof :

To prove that the matrix A is diagonalizable, we will use a theorem that if an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable. Now the characteristic polynomial of A is given by,

$$|\lambda I - A| = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix}$$

$$= (\lambda - a)(\lambda - d) - (bc)$$

$$= \lambda^2 + (-d - a)\lambda + (ad - bc)$$

.

So by appyling the quadratic formula, we get the roots of the characteristic polynomial of A. i.e.

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If we solve for $b^2 - 4ac$ we get,

$$b^2 - 4ac = (-d - a)^2 - 4(ad - bc)$$

$$= d^2 + 2ad + a^2 - 4ad + 4bc$$

$$= d^2 - 2ad + a^2 + 4bc$$

$$= (a - d)^2 + 4bc$$

Now if $b^2 - 4ac > 0$ i.e. $(a-d)^2 + 4bc > 0$ then the characteristic polynomial of A has 2 distinct real roots which implies that A has two distinct real eigenvalues. So, A is diagonalizable if $-4bc < (a-d)^2$ while if $(a-d)^2 < -4bc$ then the characteristic polynomial has imaginary roots and A is not diagonalizable.