17) $\int (\tan x + \cot x)^2 \, dx$

Solution:

$$\begin{aligned}\int (\tan x + \cot x)^2 \, dx &= \int (\tan^2 x + 2\tan x \cot x + \cot^2 x) \, dx \\ &= \int (\sec^2 x - 1 + 2 + \csc^2 x - 1) \, dx \qquad \text{(using identities for } \tan^2 x \text{ and } \cot^2 x) \\ &= \int (\sec^2 x + \csc^2 x) \, dx \\ &= \tan x - \cot x + C\end{aligned}$$

18) $\int te^{t^2} \sin(t^2) \, dt$

Solution: Using direct substitution with $x = t^2$ and $dx = 2t \, dt$, we get:

$$\int te^{t^2} \sin(t^2) \, dt = \frac{1}{2} \int e^x \sin x \, dx$$

Using integration by parts with $u = \sin x$, $du = \cos x \, dx$, and $dv = e^x \, dx$, $v = e^x$, we get:

$$\int \frac{1}{2} e^x \sin x \, dx = \frac{1}{2} e^x \sin x - \frac{1}{2} \int e^x \cos x \, dx$$

Using integration by parts again on the remaining integral with $u_1 = \cos x$, $du_1 = -\sin x \, dx$, and $dv_1 = e^x \, dx$, $v_1 = e^x$, we get:

$$\frac{1}{2} \int e^x \cos x \, dx = \frac{1}{2} e^x \cos x + \frac{1}{2} \int e^x \sin x \, dx$$

Thus,

$$\begin{aligned}\int \frac{1}{2} e^x \sin x \, dx &= \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x - \frac{1}{2} \int e^x \sin x \, dx \\ \Rightarrow \int \frac{1}{2} e^x \sin x \, dx &= \frac{1}{4} e^x \sin x - \frac{1}{4} e^x \cos x + C\end{aligned}$$

Therefore,

$$\int te^{t^2} \sin(t^2) \, dt = \frac{1}{4} e^{t^2} \sin(t^2) - \frac{1}{4} e^{t^2} \cos(t^2) + C$$

19) $\displaystyle\int \frac{2p-4}{p^2-p}\,dp$

Solution: Using partial fraction, we get:

$$\frac{2p-4}{p(p-1)} = \frac{A}{p} + \frac{B}{p-1} = \frac{A(p-1)+Bp}{p(p-1)} = \frac{(A+B)p+(-A)}{p(p-1)}$$

Thus, $A+B=2$ and $-A=-4$. So, $A=4$, and $B=-2$. We have that:

$$\int \frac{2p-4}{p(p-1)}\,dp = \int \frac{4}{p}\,dp - \int \frac{2}{p-1}\,dp$$
$$\Rightarrow \int \frac{2p-4}{p(p-1)}\,dp = 4\ln|p| - 2\ln|p-1| + C$$

20) $\displaystyle\int_3^4 \frac{1}{(3x-7)^2}\,dx$

Solution: Using direct substitution with $u=3x-7$, and $du = 3\,dx$, when $x=3$, then $u=2$, and when $x=4$, $u=5$. We have that:

$$\int_3^4 \frac{1}{(3x-7)^2}\,dx = \int_2^5 \frac{1}{3u^2}\,du = \frac{-1}{3u}\Big|_2^5 = -\frac{1}{15} + \frac{1}{6} = \frac{1}{10}$$
$$\Rightarrow \int_3^4 \frac{1}{(3x-7)^2}\,dx = \frac{1}{10}$$

21) $\displaystyle\int \frac{t^3}{(2-t^2)^{\frac{5}{2}}}\,dt$

Solution: Using direct substitution with $u = 2-t^2$, and $du = -2t\,dt$, we get:

$$\int \frac{t^3}{(2-t^2)^{\frac{5}{2}}}\,dt = \int \frac{t^2}{(2-t^2)^{\frac{5}{2}}}\,(t\,dt) = \int -\frac{2-u}{2u^{\frac{5}{2}}}\,du$$
$$= \int (-u^{-\frac{5}{2}} + \frac{1}{2}u^{-\frac{3}{2}})\,du$$
$$= \frac{2}{3}u^{-\frac{3}{2}} - u^{-\frac{1}{2}} + C$$
$$\Rightarrow \int \frac{t^3}{(2-t^2)^{\frac{5}{2}}}\,dt = \frac{2}{3}(2-t^2)^{-\frac{3}{2}} - (2-t^2)^{-\frac{1}{2}} + C$$

On the remaining integral, using direct substitution with $u = x^2 + 1$ and $du = 2x\,dx$, we get:

$$\int \frac{x}{x^2+1}\,dx = \int \frac{1}{2u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C$$

Therefore,

$$\int \frac{1}{x^3+x}\,dx = \ln|x| - \frac{1}{2}\ln(x^2+1) + C$$

Remark: This involves partial fractions with non-linear factors, which you are not required to master in this course!

29) $\displaystyle\int \ln(1+t)\,dt$

Solution: Using direct substitution with $s = 1+t$, and $ds = dt$, we have that:

$$\int \ln(1+t)\,dt = \int \ln s\,ds$$

Using integration by parts with $u = \ln s$, $du = \frac{1}{s}\,ds$, and $dv = ds$, $v = s$, we get

$$\int \ln s\,ds = s\ln s - \int s\frac{1}{s}\,ds = s\ln s - \int ds = s\ln s - s + C$$

Therefore,

$$\int \ln(1+t)\,dt = (1+t)\ln(1+t) - (1+t) + C$$

30) $\displaystyle\int \sin(3x)\cos(5x)\,dx$

Solution: Using the trigonometric identity that $\sin a\cos b = \frac{1}{2}(\sin(a+b) + \sin(a-b))$, we get:

$$\int \sin(3x)\cos(5x)\,dx = \int \frac{1}{2}(\sin(8x) + \sin(-2x))\,dx = -\frac{1}{16}\cos(8x) + \frac{1}{4}\cos(-2x) + C$$

Remark: You are not required to memorize any sum to product or product to sum trigonometric identities!