Parametric differentiation

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Example: A curve has parametric equations $x = t^2 + t, \quad y = t^3 - 3t$.

(i) Find the equation of the normal at the point where $t = 2$.

(ii) Find the points with zero gradient.

Solution: (i) When $t = 2$, $x = 6$ and $y = 2$.

$$\frac{dy}{dt} = 3t^2 - 3 \text{ and } \frac{dx}{dt} = 2t + 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t + 1} = \frac{9}{5} \quad \text{when } t = 2$$

Thus the gradient of the normal at the point $(6, 2)$ is $^{-5}/_9$

and its equation is $y - 2 = {}^{-5}/_9\,(x - 6) \Rightarrow 5x + 9y = 48$.

(ii) gradient = 0 when $\frac{dy}{dx} = \frac{3t^2 - 3}{2t + 1} = 0$

$\Rightarrow \quad 3t^2 - 3 = 0$

$\Rightarrow \quad t = \pm 1$

$\Rightarrow$ points with zero gradient are $(0, 2)$ and $(2, -2)$.

Exponential functions, a^x

$$y = a^x$$

$$\Rightarrow \quad \ln y = \ln a^x = x \ln a$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \ln a \quad \Rightarrow \quad \frac{dy}{dx} = y \ln a$$

$$\Rightarrow \quad \frac{d}{dx}\left(a^x\right) = a^x \ln a$$

5 Integration

Integrals of e^x and $\frac{1}{x}$

$$\int e^x \, dx = e^x + c$$

$$\int \frac{1}{x} \, dx = \ln|x| + c$$

for a further treatment of this result, see the appendix

Example: Find $\displaystyle\int \frac{x^3 + 3x}{x^2} \, dx$

Solution: $\displaystyle\int \frac{x^3 + 3x}{x^2} \, dx = \int x + \frac{3}{x} \, dx = \tfrac{1}{2}x^2 + 3\ln x + c.$

Standard integrals

x must be in RADIANS when integrating trigonometric functions.

$f(x)$	$\int f(x)\, dx$	$f(x)$	$\int f(x)\, dx$
x^n	$\frac{x^{n+1}}{n+1}$	$\sin x$	$-\cos x$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\cos x$	$\sin x$
e^x	e^x	$\sec x \tan x$	$\sec x$
		$\sec^2 x$	$\tan x$
		$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x$
		$\operatorname{cosec}^2 x$	$-\cot x$

Integration using trigonometric identities

Example: Find $\int \cot^2 x \, dx$.

Solution: $\cot^2 x = \operatorname{cosec}^2 x - 1$

$$\Rightarrow \int \cot^2 x \, dx = \int \operatorname{cosec}^2 x - 1 \, dx$$

$$= -\cot x - x + c.$$

and similarly

$$\int \operatorname{cosec} x \, dx = -\ln \left| \operatorname{cosec} x + \cot x \right| + c$$

Integration using partial fractions

For use with algebraic fractions where the denominator factorises.

Example: Find $\int \frac{6x}{x^2 + x - 2} \, dx$

Solution: First express $\frac{6x}{x^2 + x - 2}$ in partial fractions.

$$\frac{6x}{x^2 + x - 2} \equiv \frac{6x}{(x-1)(x+2)} \equiv \frac{A}{x-1} + \frac{B}{x+2}$$

$\Rightarrow$ $6x \equiv A(x+2) + B(x-1)$.

put $x = 1 \quad \Rightarrow \quad A = 2$,

put $x = -2 \quad \Rightarrow \quad B = 4$

$$\Rightarrow \int \frac{6x}{x^2 + x - 2} \, dx = \int \frac{2}{x-1} + \frac{4}{x+2} \, dx$$

$$= 2 \ln |x-1| + 4 \ln |x+2| + c.$$

Integration by substitution, indefinite

(i) Use the given substitution involving x and u, (or find a suitable substitution).

(ii) Find **either** $\frac{du}{dx}$ **or** $\frac{dx}{du}$, whichever is easier and re-arrange to find dx in terms of du, i.e $dx = \ldots\ldots \, du$

(iii) Use the substitution in (i) to make the integrand a function of u, and use your answer to (ii) to replace dx by $\ldots\ldots \, du$.

(iv) Simplify and integrate the function of u.

(v) Use the substitution in (i) to write your answer in terms of x.

Example: Find $\int x\sqrt{3x^2 - 5} \, dx$ using the substitution $u = 3x^2 - 5$.

Solution: (i) $u = 3x^2 - 5$

(ii) $\frac{du}{dx} = 6x \quad \Rightarrow \quad dx = \frac{du}{6x}$

(iii) We can see that there an x will cancel, and $\sqrt{3x^2 - 5} = \sqrt{u}$

$$\int x\sqrt{3x^2 - 5} \, dx = \int x\sqrt{u} \, \frac{du}{6x} = \int \frac{\sqrt{u}}{6} \, du$$

(iv) $$= \frac{1}{6}\int u^{\frac{1}{2}} \, du = \frac{1}{6} \times \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$\Rightarrow$ The equation of a straight line through the point A and parallel to the vector $\underline{\boldsymbol{b}}$ is

$\underline{\boldsymbol{r}} = \underline{\boldsymbol{a}} + \lambda\,\underline{\boldsymbol{b}}$.

Example: Find the vector equation of the line through the points M, (2, –1, 4), and N, (–5, 3, 7).

Solution: We are looking for the line through M (or N) which is parallel to the vector $\overrightarrow{MN}$.

$$\overrightarrow{MN} = \underline{\boldsymbol{n}} - \underline{\boldsymbol{m}} = \begin{bmatrix} -5 \\ 3 \\ 7 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}$$

$\Rightarrow$ equation is $\underline{\boldsymbol{r}} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}$.

Example: Show that the point P, (–1, 7, 10), lies on the line

$$\underline{\boldsymbol{r}} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}.$$

Solution: The x co-ord of P is –1 and of the line is $1 - \lambda$

$\Rightarrow$ $-1 = 1 - \lambda \Rightarrow \lambda = 2$.

In the equation of the line this gives $y = 7$ and $z = 10$

$\Rightarrow$ P, (–1, 7, 10) does lie on the line.

Intersection of two lines

2 Dimensions

Example: Find the intersection of the lines

ℓ_1, $\underline{\boldsymbol{r}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and ℓ_2, $\underline{\boldsymbol{r}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution: We are looking for values of λ and μ which give the same x and y co-ordinates on each line.

Equating x co-ords $\Rightarrow$ $2 - \lambda = 1 + \mu$

equating y co-ords $\Rightarrow$ $\underline{3 + 2\lambda = 3 - \mu}$

Adding $\Rightarrow$ $5 + \lambda = 4$ $\Rightarrow$ $\lambda = -1$ $\Rightarrow$ $\mu = 2$

$\Rightarrow$ lines intersect at $(3, 1)$.

3 Dimensions

This is similar to the method for 2 dimensions with one important difference – you can **not** be certain whether the lines intersect without checking.

You will always (or nearly always) be able to find values of λ and μ by equating x coordinates and y coordinates but the **z coordinates might or might not be equal and must be checked.**

Example: Investigate whether the lines

$$\ell_1, \quad \mathbf{\underline{r}} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \ell_2, \quad \mathbf{\underline{r}} = \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad \text{intersect}$$

and if they do find their point of intersection.

Solution: **If** the lines intersect we can find values of λ and μ to give the same x, y and z coordinates in each equation.

Equating x coords $\Rightarrow$ $2 - \lambda = -3 + \mu$, **I**

equating y coords $\Rightarrow$ $1 + 2\lambda = 1 + 3\mu$, **II**

equating z coords $\Rightarrow$ $3 + \lambda = 5 + \mu$. **III**

$2 \times \mathbf{I} + \mathbf{II} \Rightarrow 5 = -5 + 5\mu \Rightarrow \mu = 2$, in $\mathbf{I} \Rightarrow \lambda = 3$.

We must now check to see if we get the same point for the values of λ and μ

In ℓ_1, $\lambda = 3$ gives the point $(-1, 7, 6)$;

in ℓ_2, $\mu = 2$ gives the point $(-1, 7, 7)$.

The x and y co-ords are equal (as expected!), but the z co-ordinates are different and so the lines do **not** intersect.