There are also some surprising ways to use the theorem. For example, let $n \in \mathbf{Z}^+$, and let $0 \leq m \leq n$. For any positive integer k, n^k can be expressed as a sum of powers of m and $n-m$. To see this, simply note that, by the Binomial Theorem,

$$n^k = \sum_{j=0}^{k} \binom{k}{j} m^k (n-m)^{k-j}.$$

For an example, $5^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} 3^k 2^{n-k}$.

Here are some additional examples of combinatorial proof.

Example 4: A nameless algebraic identity states that

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

Here is a combinatorial solution. Its use of color is just one of several ways to differentiate the elements of the two subsets introduced to drive the proof.

Proof: The expression on the left-hand side is the number of 2-subsets of a $2n$-set. Let A be a $2n$-set, and suppose that A contains n red elements and n blue elements. We now choose all the possible 2-subsets, by counting all the choices: all the 2-subsets that have exactly 2 red elements, all the 2-subsets that have exactly 2 blue elements, and all the 2-subsets that have exactly one red element and one blue element. There are $\binom{n}{2}$ red 2-subsets, $\binom{n}{2}$ blue 2-subsets, and $\binom{n}{1}\binom{n}{1} = n^2$ subsets containing one red and one blue elements. By the Sum Rule, the number of 2-subsets of A is $2\binom{n}{2} + n^2$. $\square$

Example 5: Here is a variation on the theme. Suppose we want to prove the identity,

$$\binom{2n}{3} = 2\binom{n}{3} + 2n\binom{n}{2}.$$

The same technique used in the preceding problem leads to the following argument.

Proof: The expression on the left counts the number of 3-subsets of a $2n$-set. Let A be a $2n$-set containing n red and n blue elements. There are $\binom{n}{3}$ red 3-subsets, $\binom{n}{3}$ blue 3-subsets, $\binom{n}{2}\binom{n}{1}$ 3-subsets with two red elements and one blue, and $\binom{n}{2}\binom{n}{1}$ 3-subsets with two blue elements and one red. Simplifying, we see that the number of 3-subsets of A is given by $\binom{n}{3} + \binom{n}{3} + \binom{n}{2}\binom{n}{1} + \binom{n}{2}\binom{n}{1} = 2\binom{n}{3} + 2n\binom{n}{2}$. The result follows. $\square$

Example 6: Here's another, asking for a proof of the identity

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$