MATH 221
FIRST SEMESTER
CALCULUS

fall 2009

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Not all fractions can be represented as decimal fractions. For instance, expanding $\frac{1}{3}$ into a decimal fraction leads to an unending decimal fraction

$$\frac{1}{3} = 0.333\,333\,333\,333\,333\,\cdots$$

It is impossible to write the complete decimal expansion of $\frac{1}{3}$ because it contains infinitely many digits. But we can describe the expansion: each digit is a three. An electronic calculator, which always represents numbers as ***finite*** decimal numbers, can never hold the number $\frac{1}{3}$ exactly.

Every fraction can be written as a decimal fraction which may or may not be finite. If the decimal expansion doesn't end, then it must repeat. For instance,

$$\frac{1}{7} = 0.142857\,142857\,142857\,142857\,\ldots$$

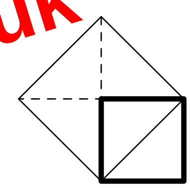
Conversely, any infinite repeating decimal expansion represents a rational number.

A ***real number*** is specified by a possibly unending decimal expansion. For instance,

$$\sqrt{2} = 1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\,698\,078\,569\,671\,875\,376\,9\ldots$$

Of course you can never write *all* the digits in the decimal expansion, so you only write the first few digits and hide the others behind dots. To give a precise description of a real number (such as $\sqrt{2}$) you have to explain how you could *in principle* compute as many digits in the expansion as you would like. During the next three semesters of calculus we will not go into the details of how this should be done.

1.2. A reason to believe in $\sqrt{2}$. The Pythagorean theorem says that the hypotenuse of a right triangle with sides 1 and 1 must be a line segment of length $\sqrt{2}$. In middle or high school you learned something similar to the following geometric construction of a line segment whose length is $\sqrt{2}$. Take a square with side of length 1, and construct a new square one of whose sides is the diagonal of the first square. The figure you get consists of 5 triangles of equal area and by counting triangles you see that the larger square has exactly twice the area of the smaller square. Therefore the diagonal of the smaller square, being the side of the larger square, is $\sqrt{2}$ as long as the side of the smaller square.



Why are real numbers called real? All the numbers we will use in this first semester of calculus are "real numbers." At some point (in 2nd semester calculus) it becomes useful to assume that there is a number whose square is -1. No real number has this property since the square of any real number is positive, so it was decided to call this new imagined number "imaginary" and to refer to the numbers we already have (rationals, $\sqrt{2}$-like things) as "real."

1.3. The real number line and intervals. It is customary to visualize the real numbers as points on a straight line. We imagine a line, and choose one point on this line, which we call the ***origin***. We also decide which direction we call "left" and hence which we call "right." Some draw the number line vertically and use the words "up" and "down."

To plot any real number x one marks off a distance x from the origin, to the right (up) if $x > 0$, to the left (down) if $x < 0$.

The ***distance along the number line*** between two numbers x and y is $|x - y|$. In particular, the distance is never a negative number.

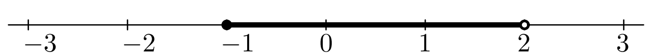


Figure 1. To draw the half open interval $[-1, 2)$ use a filled dot to mark the endpoint which is included and an open dot for an excluded endpoint.

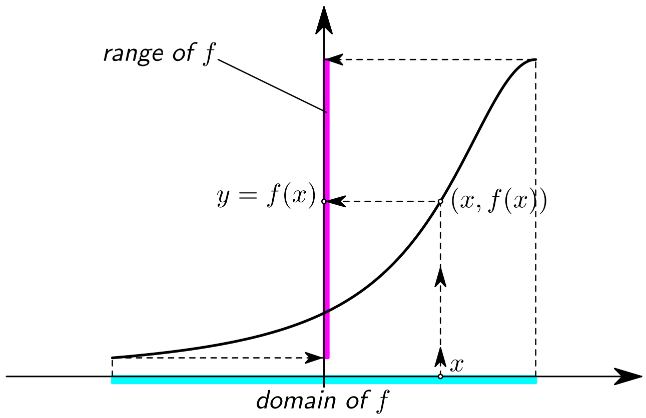


Figure 3. The graph of a function f. The domain of f consists of all x values at which the function is defined, and the range consists of all possible values f can have.

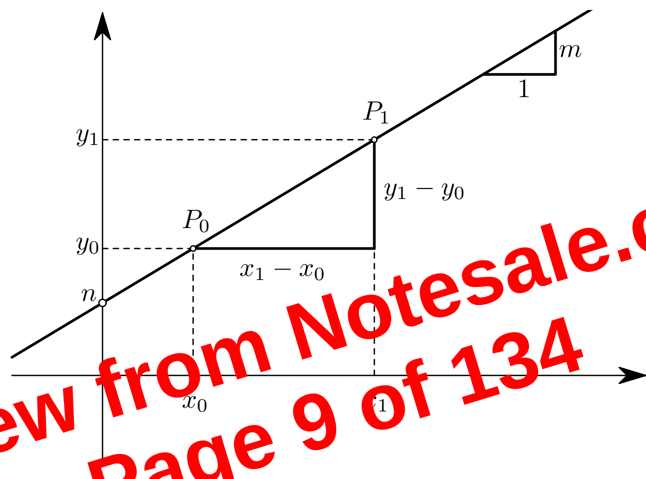


Figure 4. A straight line and its slope. The line is the graph of $f(x) = mx + n$. It intersects the y-axis at height n, and the ratio between the amounts by which y and x increase as you move from one point to another on the line is $\frac{y_1 - y_0}{x_1 - x_0} = m$.

3.3. Linear functions. A function which is given by the formula

$$f(x) = mx + n$$

where m and n are constants is called a ***linear function***. Its graph is a straight line. The constants m and n are the ***slope*** and ***y-intercept*** of the line. Conversely, any straight line which is not vertical (i.e. not parallel to the y-axis) is the graph of a linear function. If you know two points (x_0, y_0) and (x_1, y_1) on the line, then then one can compute the slope m from the "rise-over-run" formula

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

This formula actually contains a theorem from Euclidean geometry, namely it says that the ratio $(y_1 - y_0) : (x_1 - x_0)$ is the same for every pair of points (x_0, y_0) and (x_1, y_1) that you could pick on the line.

3.4. Domain and "biggest possible domain. " In this course we will usually not be careful about specifying the domain of the function. When this happens the domain is understood to be the set of all x for which the rule which tells you how to compute $f(x)$ is meaningful. For instance, if we say that h is the function

$$h(x) = \sqrt{x}$$

11.2. Example: compute $\lim_{x\to 3}\sqrt{x^3-3x^2+2}$. The given function is the composition of two functions, namely

$$\sqrt{x^3-3x^2+2}=\sqrt{u}, \text{ with } u=x^3-3x^2+2,$$

or, in function notation, we want to find $\lim_{x\to 3} h(x)$ where

$$h(x)=f(g(x)), \text{ with } g(x)=x^3-3x^2+2 \text{ and } g(x)=\sqrt{x}.$$

Either way, we have

$$\lim_{x\to 3} x^3-3x^2+2=2 \quad\text{and}\quad \lim_{u\to 2}\sqrt{u}=\sqrt{2}.$$

You get the first limit from the limit properties $(P_1)\ldots(P_5)$. The second limit says that taking the square root is a continuous function, which it is. We have not proved that (yet), but this particular limit is the one from example 6.3. Putting these two limits together we conclude that the limit is $\sqrt{2}$.

Normally, you write this whole argument as follows:

$$\lim_{x\to 3}\sqrt{x^3-3x^2+2}=\sqrt{\lim_{x\to 3} x^3-3x^2+2}=\sqrt{2},$$

where you must point out that $f(x)=\sqrt{x}$ is a continuous function to justify the first step.

Another possible way of writing this is

$$\lim_{x\to 3}\sqrt{x^3-3x^2+2}=\lim_{u\to 2}\sqrt{u}=\sqrt{2},$$

where you must say that you have substituted $u=x^3-3x^2+2$.

12. Exercises

Find the following limits.

52. $\displaystyle\lim_{x\to -7}(2x+5)$

53. $\displaystyle\lim_{x\to 7-}(2x+5)$

54. $\displaystyle\lim_{x\to -\infty}(2x+5)$

55. $\displaystyle\lim_{x\to -4}(x+3)^{2006}$

56. $\displaystyle\lim_{x\to -4}(x+3)^{2007}$

57. $\displaystyle\lim_{x\to -\infty}(x+3)^{2007}$

58. $\displaystyle\lim_{t\to 1}\frac{t^2+t-2}{t^2-1}$

59. $\displaystyle\lim_{t\nearrow 1}\frac{t^2+t-2}{t^2-1}$

60. $\displaystyle\lim_{t\to -1}\frac{t^2+t-2}{t^2-1}$

61. $\displaystyle\lim_{x\to\infty}\frac{x^2+3}{x^2+4}$

62. $\displaystyle\lim_{x\to\infty}\frac{x^5+3}{x^2+4}$

63. $\displaystyle\lim_{x\to\infty}\frac{x^2+1}{x^5+2}$

64. $\displaystyle\lim_{x\to\infty}\frac{(2x+1)^4}{(3x^2+1)^2}$

65. $\displaystyle\lim_{u\to\infty}\frac{(2u+1)^4}{(3u^2+1)^2}$

66. $\displaystyle\lim_{t\to 0}\frac{(2t+1)^4}{(3t^2+1)^2}$

67. What are the coordinates of the points labeled A, …, E in Figure 2 (the graph of $y=\sin\pi/x$).

68. If $\lim_{x\to a} f(x)$ exists then f is continuous at $x=a$. *True or false?*

69. Give two examples of functions for which $\lim_{x\searrow 0} f(x)$ does not exist.

70. Group Problem.

If $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ both do not exist, then $\lim_{x\to 0}\big(f(x)+g(x)\big)$ also does not exist. *True or false?*

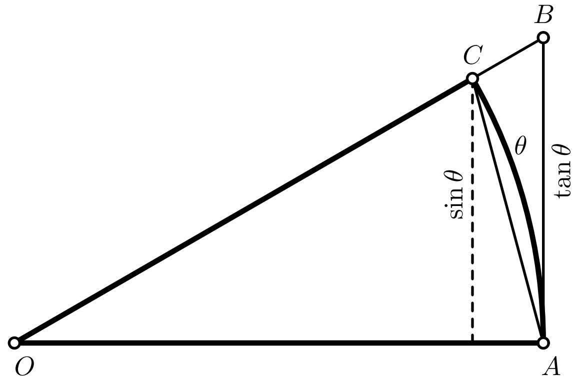
71. Group Problem.

If $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ both do not exist, then $\lim_{x\to 0}(f(x)/g(x))$ also does not exist. *True or false?*

72. Group Problem.

In the text we proved that $\lim_{x\to\infty}\frac{1}{x}=0$. Show that this implies that $\lim_{x\to\infty} x$ does not exist. Hint: Suppose $\lim_{x\to\infty} x=L$ for some number L. Apply the limit properties to $\lim_{x\to\infty} x\cdot\dfrac{1}{x}$.

73. Evaluate $\displaystyle\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$. Hint: Multiply top and bottom by $\sqrt{x}+3$.



Proving $\lim_{x\to 0} \frac{\sin\theta}{\theta} = 1$

The circular wedge OAC contains the triangle OAC and is contained in the right triangle OAB.

The area of triangle OAC is $\frac{1}{2}\sin\theta$.

The area of circular wedge OAC is $\frac{1}{2}\theta$.

The area of right triangle OAB is $\frac{1}{2}\tan\theta$.

Hence one has $\sin\theta < \theta < \tan\theta$ for all angles $0 < \theta < \pi/2$.

74. Evaluate $\lim_{x\to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x-2}$.

75. Evaluate $\lim_{x\to 2} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{2}}}{x-2}$.

76. A function f is defined by

$$f(x) = \begin{cases} x^3 & \text{for } x < -1 \\ ax + b & \text{for } -1 \le x < 1 \\ x^2 + 2 & \text{for } x \ge 1. \end{cases}$$

where a and b are constants. The function f is continuous. What are a and b?

77. Find a constant k such that the function

$$f(x) = \begin{cases} 3x + 2 & \text{for } x < 2 \\ x^2 + k & \text{for } x \ge 2. \end{cases}$$

is continuous. Hint: Compute the one sided limits.

78. Find constants a and c such that the function

$$f(x) = \begin{cases} x^3 + c & \text{for } x < 0 \\ ax + c^2 & \text{for } 0 \le x < 1 \\ \arctan x & \text{for } x \ge 1. \end{cases}$$

is continuous for all x.

13. Two Limits in Trigonometry

In this section we'll derive a few limits involving the trigonometric functions. You can think of them as saying that for small angles θ one has

$$\sin\theta \approx \theta \quad \text{and} \quad \cos\theta \approx 1 - \tfrac{1}{2}\theta^2.$$

We will use these limits when we compute the derivatives of Sine, Cosine and Tangent.

13.1. Theorem. $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$

Proof. The proof requires a few sandwiches and some geometry.

We begin by only considering positive angles, and in fact we will only consider angles $0 < \theta < \pi/2$.

Since the wedge OAC contains the triangle OAC its area must be larger. The area of the wedge is $\frac{1}{2}\theta$ and the area of the triangle is $\frac{1}{2}\sin\theta$, so we find that

$$0 < \sin\theta < \theta \text{ for } 0 < \theta < \frac{\pi}{2}. \tag{13}$$

The Sandwich Theorem implies that

$$\lim_{\theta\searrow 0} \sin\theta = 0. \tag{14}$$

Moreover, we also have

$$\lim_{\theta\searrow 0} \cos\theta = \lim_{\theta\searrow 0} \sqrt{1 - \sin^2\theta} = 1. \tag{15}$$

97. Is there a constant k such that the function

$$f(x) = \begin{cases} \sin(1/x) & \text{for } x \neq 0 \\ k & \text{for } x = 0. \end{cases}$$

is continuous? If so, find it; if not, say why.

98. Find a constant A so that the function

$$f(x) = \begin{cases} \dfrac{\sin x}{2x} & \text{for } x \neq 0 \\ A & \text{when } x = 0 \end{cases}$$

99. Compute $\lim_{x\to\infty} x \sin\frac{\pi}{x}$ and $\lim_{x\to\infty} x \tan\frac{\pi}{x}$. (Hint: substitute something).

100. Group Problem.

(*Geometry & Trig review*) Let A_n be the area of the regular n-gon inscribed in the unit circle, and let B_n be the area of the regular n-gon whose inscribed circle has radius 1.

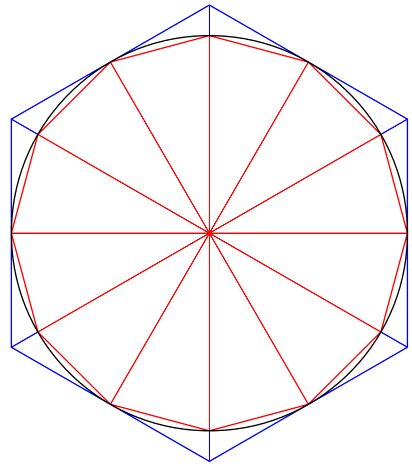
(a) Show that $A_n < \pi < B_n$.

(b) Show that

$$A_n = \frac{n}{2}\sin\frac{2\pi}{n} \text{ and } B_n = n\tan\frac{\pi}{n}$$

(c) Compute $\lim_{n\to\infty} A_n$ and $\lim_{n\to\infty} B_n$.

Here is a picture of A_{12}, B_6 and π:



On a historical note: Archimedes managed to compute A_{96} and B_{96} and by doing this got the most accurate approximation for π that was known in his time. See also:

http://www-history.mcs.st-andrews.ac.uk/HistTopics/Pi_through_the_ages.html

If you don't remember the geometric sum formula, then you could also just verify (19) by carefully multiplying both sides with $x - a$. For instance, when $n = 3$ you would get

$$\begin{array}{rcllll} x \times (x^2 + xa + a^2) & = & x^3 & +ax^2 & +a^2x & \\ -a \times (x^2 + xa + a^2) & = & & -ax^2 & -a^2x & -a^3 \\ \hline (x-a) \times (x^2 + ax + a^2) & = & x^3 & & & -a^3 \end{array}$$

With formula (19) in hand we can now easily find the derivative of x^n:

$$\begin{aligned} f'(a) &= \lim_{x\to a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x\to a} \left\{x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}\right\} \\ &= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \cdots + a\, a^{n-2} + a^{n-1}. \end{aligned}$$

Here there are n terms, and they all are equal to a^{n-1}, so the final result is

$$f'(a) = na^{n-1}.$$

One could also write this as $f'(x) = nx^{n-1}$, or, in Leibniz' notation

$$\frac{dx^n}{dx} = nx^{n-1}.$$

This formula turns out to be true in general, but here we have only proved it for the case in which n is a positive integer.

3. Differentiable implies Continuous

3.1. Theorem. *If a function f is differentiable at some a in its domain, then f is also continuous at a.*

Proof. We are given that

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

exists, and we must show that

$$\lim_{x\to a} f(x) = f(a).$$

This follows from the following computation

$$\begin{aligned} \lim_{x\to a} f(x) &= \lim_{x\to a} \big(f(x) - f(a) + f(a)\big) && \text{(algebra)} \\ &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot (x - a) + f(a) && \text{(more algebra)} \\ &= \left\{\lim_{x\to a} \frac{f(x) - f(a)}{x - a}\right\} \cdot \lim_{x\to a} (x - a) + \lim_{x\to a} f(a) && \text{(Limit Properties)} \\ &= f'(a) \cdot 0 + f(a) && (f'(a) \text{ exists}) \\ &= f(a). \end{aligned}$$

□

4. Some non-differentiable functions

4.1. A graph with a corner. Consider the function

$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0, \\ -x & \text{for } x < 0. \end{cases}$$

This function is continuous at all x, but it is not differentiable at $x = 0$.

6. The Differentiation Rules

You could go on and compute more derivatives from the definition. Each time you would have to compute a new limit, and hope that there is some trick that allows you to find that limit. This is fortunately not necessary. It turns out that if you know a few basic derivatives (such as $dx^n/dx = nx^{n-1}$) the you can find derivatives of arbitrarily complicated functions by breaking them into smaller pieces. In this section we'll look at rules which tell you how to differentiate a function which is either the sum, difference, product or quotient of two other functions.

Constant rule:	$c' = 0$	$\dfrac{dc}{dx} = 0$
Sum rule:	$(u \pm v)' = u' \pm v'$	$\dfrac{du \pm v}{dx} = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$
Product rule:	$(u \cdot v)' = u' \cdot v + u \cdot v'$	$\dfrac{duv}{dx} = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$
Quotient rule:	$\left(\dfrac{u}{v}\right)' = \dfrac{u' \cdot v - u \cdot v'}{v^2}$	$\dfrac{d\dfrac{u}{v}}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Table 1. The differentiation rules

The situation is analogous to that of the "limit properties" $(P_1)\ldots(P_6)$ from the previous chapter which allowed us to compute limits without always having to go back to the epsilon-delta definition.

6.1. Sum, product and quotient rules. In the following c and n are constants, u and v are functions of x, and $'$ denotes differentiation. The Differentiation Rules in function notation, and Leibniz notation, are listed in figure 1.

Note that we already proved the Constant Rule in example 2.2. We will now prove the sum, product and quotient rules.

6.2. Proof of the Sum Rule. Suppose that $f(x) = u(x)+v(x)$ for all x where u and v are differentiable. Then

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} && \text{(definition of } f'\text{)}\\
&= \lim_{x\to a} \frac{\big(u(x) + v(x)\big) - \big(u(a) + v(a)\big)}{x - a} && \text{(use } f = u + v\text{)}\\
&= \lim_{x\to a} \left(\frac{u(x) - u(a)}{x - a} + \frac{v(x) - v(a)}{x - a}\right) && \text{(algebra)}\\
&= \lim_{x\to a} \frac{u(x) - u(a)}{x - a} + \lim_{x\to a} \frac{v(x) - v(a)}{x - a} && \text{(limit property)}\\
&= u'(a) + v'(a) && \text{(definition of } u',\ v'\text{)}
\end{aligned}$$

6.3. Proof of the Product Rule. Let $f(x) = u(x)v(x)$. To find the derivative we must express the change of f in terms of the changes of u and v

$$\begin{aligned}
f(x) - f(a) &= u(x)v(x) - u(a)v(a)\\
&= u(x)v(x) - u(x)v(a) + u(x)v(a) - u(a)v(a)\\
&= u(x)\big(v(x) - v(a)\big) + \big(u(x) - u(a)\big)v(a)
\end{aligned}$$

Since g is a differentiable function it must also be a continuous function, and hence $\lim_{x\to a} g(x) = g(a)$. So we can substitute $y = g(x)$ in the limit defining $f'(g(a))$

$$f'(g(a)) = \lim_{y\to a} \frac{f(y) - f(g(a))}{y - g(a)} = \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)}. \tag{27}$$

Put all this together and you get

$$\begin{aligned}
(f\circ g)'(a) &= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x - a} \\
&= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \\
&= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \lim_{x\to a} \frac{g(x) - g(a)}{x - a} \\
&= f'(g(a)) \ \cdot\ g'(a)
\end{aligned}$$

which is what we were supposed to prove – the proof seems complete.

There is one flaw in this proof, namely, we have divided by $g(x) - g(a)$, which is not allowed when $g(x) - g(a) = 0$. This flaw can be fixed but we will not go into the details here.[2] □

13.3. First example. We go back to the functions

$$z = f(y) = y^2 + y \text{ and } y = g(x) = 2x + 1$$

from the beginning of this section. The composition of these two functions is

$$z = f(g(x)) = (2x+1)^2 + (2x+1) = 4x^2 + 6x + 2.$$

We can compute the derivative of this composed function, i.e. the derivative of z with respect to x in two ways. First, you simply differentiate the last formula we have:

$$\frac{dz}{dx} = \frac{d(4x^2 + 6x + 2)}{dx} = 8x + 6. \tag{28}$$

The other approach is to use the chain rule:

$$\frac{dz}{dy} = \frac{d(y^2 + y)}{dy} = 2y + 1,$$

and

$$\frac{dy}{dx} = \frac{d(2x + 1)}{dx} = 2.$$

Hence, by the chain rule one has

$$\frac{dz}{dx} = \frac{dz}{dy}\,\frac{dy}{dx} = (2y + 1)\cdot 2 = 4y + 2. \tag{29}$$

The two answers (28) and (29) should be the same. Once you remember that $y = 2x + 1$ you see that this is indeed true:

$$y = 2x + 1 \implies 4y + 2 = 4(2x + 1) + 2 = 8x + 6.$$

The two computations of dz/dx therefore lead to the same answer. In this example there was no clear advantage in using the chain rule. The chain rule becomes useful when the functions f and g become more complicated.

[2] Briefly, you have to show that the function

$$h(y) = \begin{cases} \{f(y) - f(g(a))\}/(y - g(a)) & y \neq a \\ f'(g(a)) & y = a \end{cases}$$

is continuous.

The direct approach goes like this:

$$\begin{aligned} f'(x) &= \frac{d\big(1-x^4\big)^{1/4}}{dx} \\ &= \tfrac{1}{4}\big(1-x^4\big)^{-3/4}\frac{d(1-x^4)}{dx} \\ &= \tfrac{1}{4}\big(1-x^4\big)^{-3/4}\big(-4x^3\big) \\ &= -\frac{x^3}{\big(1-x^4\big)^{3/4}} \end{aligned}$$

To find the derivative using implicit differentiation we must first find a nice implicit description of the function. For instance, we could decide to get rid of all roots or fractional exponents in the function and point out that $y = \sqrt[4]{1-x^4}$ satisfies the equation $y^4 = 1 - x^4$. So our implicit description of the function $y = f(x) = \sqrt[4]{1-x^4}$ is

$$x^4 + y^4 - 1 = 0; \qquad \text{The defining function is therefore } F(x,y) = x^4 + y^4 - 1$$

Differentiate both sides with respect to x (and remember that $y = f(x)$, so y here is a function of x), and you get

$$\frac{dx^4}{dx} + \frac{dy^4}{dx} - \frac{d1}{dx} = 0 \implies 4x^3 + 4y^3\frac{dy}{dx} = 0.$$

The expressions G and H from equation (30) in the recipe are $G(x,y) = 4y^3$ and $H(x,y) = 4x^3$.

This last equation can be solved for dy/dx:

$$\frac{dy}{dx} = -\frac{x^3}{y^3}.$$

This is a nice and short form of the derivative, but it contains y as well as x. To express dy/dx in terms of x only, and remove the y dependency we use $y = \sqrt[4]{1-x^4}$. The result is

$$f'(x) = \frac{dy}{dx} = -\frac{x^3}{y^3} = -\frac{x^3}{\big(1-x^4\big)^{3/4}}.$$

15.4. Another example. Let f be a function defined by

$$y = f(x) \iff 2y + \sin y = x, \text{ i.e. } 2y + \sin y - x = 0.$$

For instance, if $x = 2\pi$ then $y = \pi$, i.e. $f(2\pi) = \pi$.

To find the derivative dy/dx we differentiate the defining equation

$$\frac{d(2y + \sin y - x)}{dx} = \frac{d0}{dx} \implies 2\frac{dy}{dx} + \cos y\frac{dy}{dx} - \frac{dx}{dx} = 0 \implies (2 + \cos y)\frac{dy}{dx} - 1 = 0.$$

Solve for $\frac{dy}{dx}$ and you get

$$f'(x) = \frac{1}{2 + \cos y} = \frac{1}{2 + \cos f(x)}.$$

If we were asked to find $f'(2\pi)$ then, since we know $f(2\pi) = \pi$, we could answer

$$f'(2\pi) = \frac{1}{2 + \cos \pi} = \frac{1}{2-1} = 1.$$

If we were asked $f'(\pi/2)$, then all we would be able to say is

$$f'(\pi/2) = \frac{1}{2 + \cos f(\pi/2)}.$$

To say more we would first have to find $y = f(\pi/2)$, which one does by solving

$$2y + \sin y = \frac{\pi}{2}.$$

15.5. Derivatives of Arc Sine and Arc Tangent. Recall that

$$y = \arcsin x \iff x = \sin y \text{ and } -\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2},$$

and

$$y = \arctan x \iff x = \tan y \text{ and } -\tfrac{\pi}{2} < y < \tfrac{\pi}{2}.$$

15.6. Theorem.

$$\frac{d \arcsin x}{dx} = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d \arctan x}{dx} = \frac{1}{1+x^2}$$

PROOF. If $y = \arcsin x$ then $x = \sin y$. Differentiate this relation

$$\frac{dx}{dx} = \frac{d \sin y}{dx}$$

and apply the chain rule. You get

$$1 = \left(\cos y\right) \frac{dy}{dx},$$

and hence

$$\frac{dy}{dx} = \frac{1}{\cos y}.$$

How do we get rid of the y on the right hand side? We know $x = \sin y$, and also $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Therefore

$$\sin^2 y + \cos^2 y = 1 \implies \cos y = \pm\sqrt{1-\sin^2 y} = \pm\sqrt{1-x^2}.$$

Since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ we know that $\cos y \ge 0$, so we must choose the positive square root. This leaves us with $\cos y = \sqrt{1-x^2}$, and hence

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

The derivative of $\arctan x$ is found in the same way, and you should really do this yourself. □

16. Exercises

For each of the following problems find the derivative $f'(x)$ if $y = f(x)$ satisfies the given equation. State what the expressions $F(x,y)$, $G(x,y)$ and $H(x,y)$ from the recipe in the beginning of this section are.

If you can find an explicit description of the function $y = f(x)$, say what it is.

167. $xy = \dfrac{\pi}{6}$

168. $\sin(xy) = \frac{1}{2}$

169. $\dfrac{xy}{x+y} = 1$

170. $x + y = xy$

171. $(y-1)^2 + x = 0$

172. $(y+1)^2 + y - x = 0$

173. $(y-x)^2 + x = 0$

174. $(y+x)^2 + 2y - x = 0$

175. $\left(y^2-1\right)^2 + x = 0$

176. $\left(y^2+1\right)^2 - x = 0$

177. $x^3 + xy + y^3 = 3$

178. $\sin x + \sin y = 1$

179. $\sin x + xy + y^5 = \pi$

180. $\tan x + \tan y = 1$

For each of the following explicitly defined functions find an implicit definition which does not involve taking roots. Then use this description to find the derivative dy/dx.

181. $y = f(x) = \sqrt{1-x}$

182. $y = f(x) = \sqrt[4]{x+x^2}$

183. $y = f(x) = \sqrt{1-\sqrt{x}}$

184. $y = f(x) = \sqrt[4]{x-\sqrt{x}}$

185. $y = f(x) = \sqrt[3]{\sqrt{2x+1}-x^2}$

186. $y = f(x) = \sqrt[4]{x+x^2}$

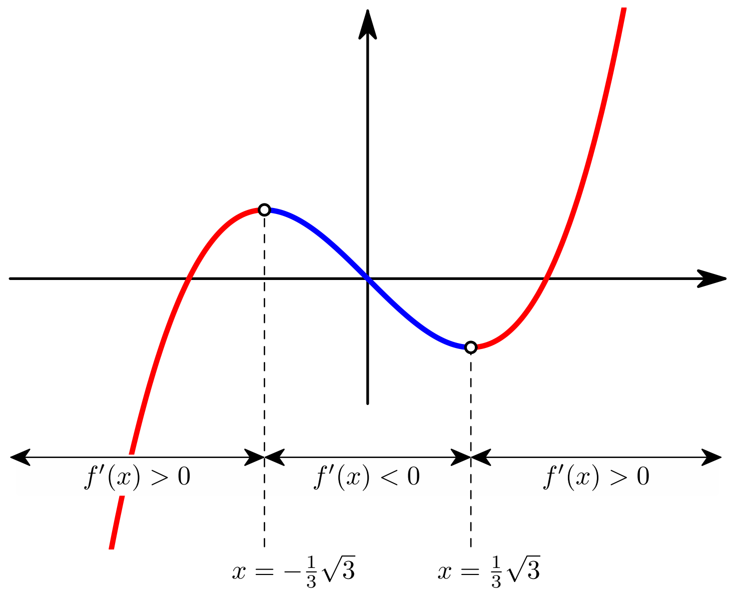


Figure 4. The graph of $f(x) = x^3 - x$.

6.4. A function whose tangent turns up and down infinitely often near the origin. We end with a weird example. Somewhere in the mathematician's zoo of curious functions the following will be on exhibit. Consider the function

$$f(x) = \frac{x}{2} + x^2 \sin \frac{\pi}{x}.$$

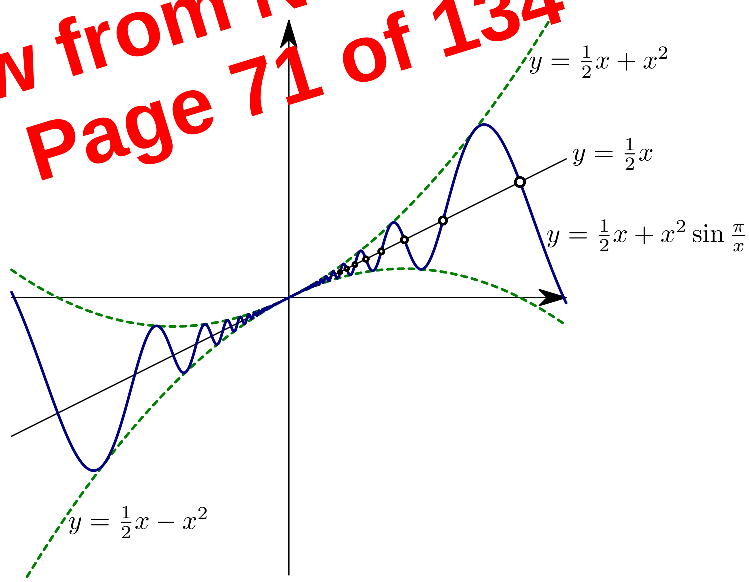


Figure 5. Positive derivative at a point ($x = 0$) does not mean that the function is "increasing near that point." The slopes at the intersection points alternate between $\frac{1}{2} - \pi$ and $\frac{1}{2} + \pi$.

For $x = 0$ this formula is undefined, and we are free to define $f(0) = 0$. This makes the function continuous at $x = 0$. In fact, this function is differentiable at $x = 0$, with derivative given by

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{1}{2} + x \sin \frac{\pi}{x} = \frac{1}{2}.$$

Figure 9. If a graph is convex then all chords lie above the graph. If it is not convex then some chords will cross the graph or lie below it.

Between the two stationary points the function is increasing, so

$$f(-1-\sqrt{2}) \le f(x) \le f(B) \text{ for } A \le x \le B.$$

From this it follows that $f(x)$ is the smallest it can be when $x = A = -1-\sqrt{2}$ and at its largest when $x = B = -1+\sqrt{2}$: the local maximum and minimum which we found are in fact a global maximum and minimum.

11. Convexity, Concavity and the Second Derivative

By definition, a function f is ***convex*** on some interval $a < x < b$ if the line segment connecting any pair of points on the graph lies *above* the piece of the graph between those two points.

The function is called ***concave*** if the line segment connecting any pair of points on the graph lies *below* the piece of the graph between those two points.

A point on the graph of f where $f''(x)$ changes sign is called an ***inflection point***.

Instead of "convex" and "concave" one often says "curved upwards" or "curved downwards."

You can use the second derivative to tell if a function is concave or convex.

11.1. Theorem. *A function f is convex on some interval $a < x < b$ if and only if $f''(x) \ge 0$ for all x on that interval.*

11.2. Theorem. *A function f is convex on some interval $a < x < b$ if and only if the derivative $f'(x)$ is a nondecreasing function on that interval.*

A proof using the Mean Value Theorem will be given in class.

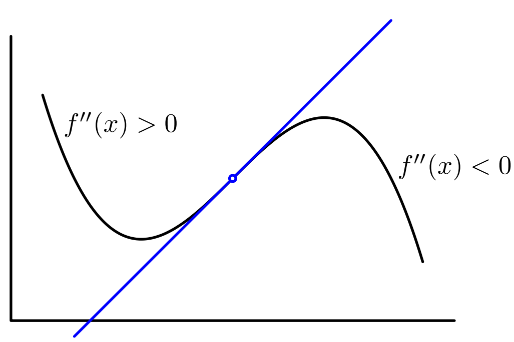


Figure 10. At an inflection point the tangent crosses the graph.

as claimed.

13. Exercises

218. What does the Intermediate Value Theorem say?

219. What does the Mean Value Theorem say?

220. Group Problem.

If $f(a) = 0$ and $f(b) = 0$ then there is a c between a and b such that $f'(c) = 0$. Show that this follows from the Mean Value Theorem. (Help! A proof! Relax: this one is not difficult. Make a drawing of the situation, then read the Mean Value Theorem again.)

221. What is a stationary point?

222. Group Problem.

How can you tell if a local maximum is a global maximum?

223. Group Problem.

If $f''(a) = 0$ then the graph of f has an inflection point at $x = a$. **True or False?**

224. What is an inflection point?

225. Give an example of a function for which $f'(0) = 0$ even though the function f has neither a local maximum or a local minimum at $x = 0$.

226. Group Problem.

Draw four graphs of functions, one for each of the following four combinations

$f' > 0$ and $f'' > 0$ $\qquad$ $f' > 0$ and $f'' < 0$

$f' < 0$ and $f'' > 0$ $\qquad$ $f' < 0$ and $f'' < 0$

227. Group Problem.

Which of the following combinations are possible:

$f'(x) > 0$ and $f''(x) = 0$ for all x

$f'(x) = 0$ and $f''(x) > 0$ for all x

Sketch the graph of the following functions. You should

(1) find where f, f' and f'' are positive or negative
(2) find all stationary points
(3) decide which stationary points are local maxima or minima
(4) decide which local max/minima are in fact global max/minima
(5) find all inflection points
(6) find "horizontal asymptotes," i.e. compute the limits $\lim_{x\to\pm\infty} f(x)$ when appropriate.

228. $y = x^3 + 2x^2$

229. $y = x^3 - 4x^2$

230. $y = x^4 + 27x$

231. $y = x^4 - 27x$

232. $y = x^4 + 2x^2 - 3$

233. $y = x^4 - 5x^2 + 4$

234. $y = x^5 + 16x$

235. $y = x^5 - 16x$

236. $y = \dfrac{x}{x+1}$

237. $y = \dfrac{x}{1+x^2}$

238. $y = \dfrac{x^2}{1+x^2}$

239. $y = \dfrac{1+x^2}{1+x}$

240. $y = x + \dfrac{1}{x}$

241. $y = x - \dfrac{1}{x}$

242. $y = x^3 + 2x^2 + x$

243. $y = x^3 + 2x^2 - x$

244. $y = x^4 - x^3 - x$

245. $y = x^4 - 2x^3 + 2x$

246. $y = \sqrt{1+x^2}$

247. $y = \sqrt{1-x^2}$

248. $y = \sqrt[4]{1+x^2}$

249. $y = \dfrac{1}{1+x^4}$

The following functions are periodic, i.e. they satisfy $f(x+L) = f(x)$ for all x, where the constant L is called the period of the function. The graph of a periodic function repeats itself indefinitely to the left and to the right. It therefore has infinitely many (local) minima and maxima, and infinitely many inflections points. Sketch the graphs of the following functions as in the previous problem, but only list those "interesting points" that lie in the interval $0 \le x < 2\pi$.

250. $y = \sin x$

251. $y = \sin x + \cos x$

252. $y = \sin x + \sin^2 x$

253. $y = 2\sin x + \sin^2 x$

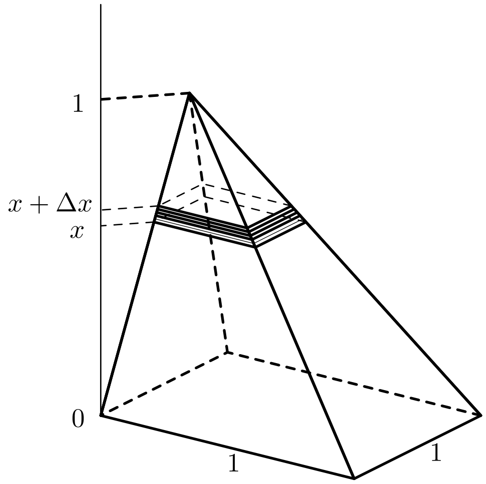


Figure 2. The slice at height x is a square with side $1 - x$.

Therefore there is some c_k in the interval $[x_{k-1}, x_k]$ such that

$$\text{volume of } k^{\text{th}} \text{ slice} = (1 - c_k)^2 \Delta x_k.$$

Adding the volumes of the slices we find that the volume V of the pyramid is given by

$$V = (1 - c_1)^2 \Delta x_1 + \cdots + (1 - c_N)^2 \Delta x_N.$$

The right hand side in this equation is a Riemann sum for the integral

$$I = \int_0^1 (1 - x)^2 dx$$

and therefore we have

$$I = \lim_{\cdots} \big\{ (1 - c_1)^2 \Delta x_1 + \cdots + (1 - c_N)^2 \Delta x_N \big\} = V.$$

Compute the integral and you find that the volume of the pyramid is

$$V = \frac{1}{3}.$$

3.2. General case. The "method of slicing" which we just used to compute the volume of a pyramid works for solids of any shape. The strategy always consists of dividing the solid into many thin (horizontal) slices, compute their volumes, and recognize that the total volume of the slices is a Riemann sum for some integral. That integral then is the volume of the solid.

To be more precise, let a and b be the heights of the lowest and highest points on the solid, and let $a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = b$ be a partition of the interval $[a, b]$. Such a partition divides the solid into N distinct slices, where slice number k consists of all points in the solid whose height is between x_{k-1} and x_k. The thickness of the k^{th} slice is $\Delta x_k = x_k - x_{k-1}$. If

$$A(x) = \text{area of the intersection of the solid with the plane at height } x.$$

then we can approximate the volume of the k^{th} slice by

$$A(c_k) \Delta x_k$$

where c_k is any number (height) between x_{k-1} and x_k.

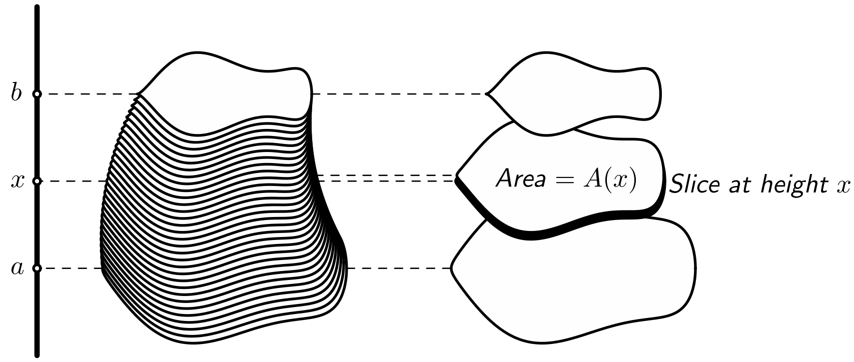


Figure 3. Slicing a solid to compute its volume. The volume of one slice is approximately the product of its thickness (Δx) and the area $A(x)$ of its top. Summing the volume $A(x)\Delta x$ over all slices leads approximately to the integral $\int_a^b f(x)dx$.

The total volume of all slices is therefore approximately

$$V \approx A(c_1)\Delta x_1 + \cdots + A(c_N)\Delta x_N.$$

While this formula only holds approximately, we expect the approximation to get better as we make the partition finer, and thus

$$V = \lim_{\cdots} \{A(c_1)\Delta x_1 + \cdots + A(c_N)\Delta x_N\}. \tag{62}$$

On the other hand the sum on the right is a Riemann sum for the integral $I = \int_a^b A(x)dx$, so the limit is exactly this integral. Therefore we have

$$V = \int_a^b A(x)dx. \tag{63}$$

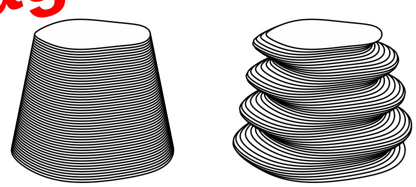


Figure 4. Cavalieri's principle. Both solids consist of a pile of horizontal slices. The solid on the right was obtained from the solid on the left by sliding some of the slices to the left and others to the right. This operation does not affect the volumes of the slices, and hence both solids have the same volume.

3.3. Cavalieri's principle. The formula (63) for the volume of a solid which we have just derived shows that the volume only depends on the areas $A(x)$ of the cross sections of the solid, and not on the particular shape these cross sections may have. This observation is older than calculus itself and goes back at least to Bonaventura Cavalieri (1598 – 1647) who said: *If the intersections of two solids with a horizontal plane always have the same area, no matter what the height of the horizontal plane may be, then the two solids have the same volume.*

This principle is often illustrated by considering a stack of coins: If you put a number of coins on top of each other then the total volume of the coins is just the sum of the volumes of the coins. If you change the shape of the pile by sliding the coins horizontally then the volume of the pile will still be the sum of the volumes of the coins, i.e. it doesn't change.

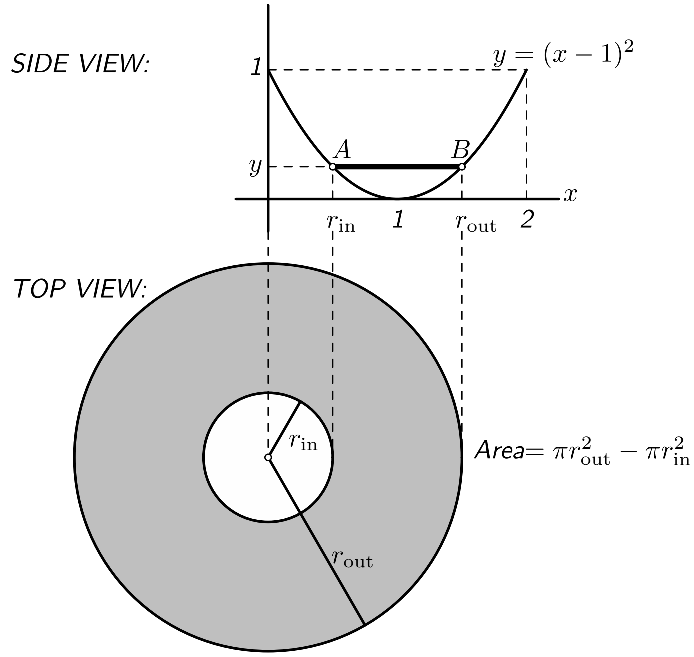
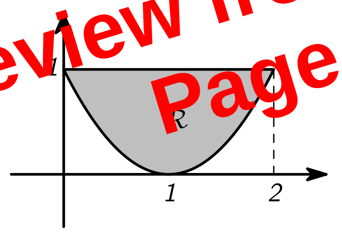
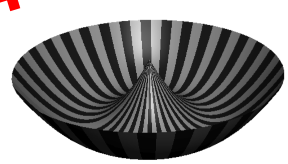


Figure 6. Computing the volume of the solid you get when you revolve the region $\mathcal{R}$ around the y-axis. A horizontal cross section of the solid is a "washer" with inner radius r_{in}, and outer radius r_{out}.





Solution: The region we have to revolve around the y-axis consists of all points above the parabola $y = (x-1)^2$ but below the line $y = 1$.

If we intersect the solid with a plane at height y then we get a ring shaped region, or "annulus", i.e. a large disc with a smaller disc removed. You can see it in the figure below: if you cut the region $\mathcal{R}$ horizontally at height y you get the line segment AB, and if you rotate this segment around the y-axis you get the grey ring region pictured below the graph. Call the radius of the outer circle r_{out} and the radius of the inner circle r_{in}. These radii are the two solutions of

$$y = (1-r)^2$$

so they are

$$r_{\text{in}} = 1 - \sqrt{y}, \qquad r_{\text{out}} = 1 + \sqrt{y}.$$

The area of the cross section is therefore given by

$$A(y) = \pi r_{\text{out}}^2 - \pi r_{\text{in}}^2 = \pi\big(1+\sqrt{y}\big)^2 - \pi\big(1-\sqrt{y}\big)^2 = 4\pi\sqrt{y}.$$

As before the slices are ring shaped regions but the inner and outer radii are now given by

$$r_{\text{in}} = 1 + x_{\text{in}} = 2 - \sqrt{y}, \qquad r_{\text{out}} = 1 + x_{\text{out}} = 2 + \sqrt{y}.$$

The volume is therefore given by

$$V = \int_0^1 \left(\pi r_{\text{out}}^2 - \pi r_{\text{in}}^2\right) dy = \pi \int_0^1 8\sqrt{y}\, dy = \frac{16\pi}{3}.$$

4.3. Problem 3: Revolve $\mathcal{R}$ around the line $y = 2$. Compute the volume of the solid you get when you revolve the same region $\mathcal{R}$ around the line $y = 2$.

Solution: This time the line around which we rotate $\mathcal{R}$ is horizontal, so we slice the solid with planes perpendicular to the x-axis.

A typical slice is obtained by revolving the line segment AB about the line $y = 2$. The result is again an annulus, and from the figure we see that the inner and outer radii of the annulus are

$$r_{\text{in}} = 1, \qquad r_{\text{out}} = 2 - (1 - x)^2.$$

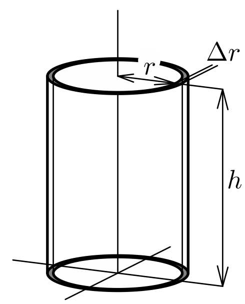
The area of the slice is therefore

$$A(x) = \pi\left\{2 - (1-x)^2\right\}^2 - \pi 1^2 = \pi\left\{3 - 4(1-x)^2 + (1-x)^4\right\}.$$

The x values which occur in the solid are $0 \le x \le 2$, and so its volume is

$$\begin{aligned} V &= \pi \int_0^2 \left\{3 - 4(1-x)^2 + (1-x)^4\right\} dx \\ &= \pi\left[3x + \tfrac{4}{3}(1-x)^3 - \tfrac{1}{5}(1-x)^5\right]_0^2 \\ &= \tfrac{56}{15}\pi \end{aligned}$$

5. Volumes by cylindrical shells



Instead of slicing a solid with planes you can also try to decompose it into cylindrical shells. The volume of a cylinder of height h and radius r is $\pi r^2 h$ (height times area base). Therefore the volume of a cylindrical shell of height h, (inner) radius r and thickness Δr is

$$\begin{aligned} \pi h(r + \Delta r)^2 - \pi h r^2 &= \pi h(2r + \Delta r)\Delta r \\ &\approx 2\pi h r \Delta r. \end{aligned}$$

Now consider the solid you get by revolving the region

$$\mathcal{R} = \{(x, y) \mid a \le x \le b, 0 \le y \le f(x)\}$$

around the y-axis. By partitioning the interval $a \le x \le b$ into many small intervals we can decompose the solid into many thin shells. The volume of each shell will approximately be given by $2\pi x f(x)\Delta x$. Adding the volumes of the shells, and taking the limit over finer and finer partitions we arrive at the following formula for the volume of the solid of revolution:

$$V = 2\pi \int_a^b x f(x)\, dx. \tag{65}$$

If the region $\mathcal{R}$ is not the region under the graph, but rather the region between the graphs of two functions $f(x) \le g(x)$, then we get

$$V = 2\pi \int_a^b x\left\{g(x) - f(x)\right\} dx.$$

11.2. Kinetic energy. Newton's famous law relating the force exerted on an object and its motion says $F = ma$, where a is the acceleration of the object, m is its mass, and F is the combination of all forces acting on the object. If the position of the object at time t is $x(t)$, then its velocity and acceleration are $v(t) = x'(t)$ and $a(t) = v'(t) = x''(t)$, and thus the total force acting on the object is

$$F(t) = ma(t) = m\frac{dv}{dt}.$$

The work done by the total force is therefore

$$W = \int_{t_a}^{t_b} F(t)v(t)dt = \int_{t_a}^{t_b} m\frac{dv(t)}{dt}\, v(t)\, dt. \tag{72}$$

Even though we have not assumed anything about the motion, so we don't know anything about the velocity $v(t)$, we can still do this integral. The key is to notice that, by the chain rule,

$$m\frac{dv(t)}{dt}\, v(t) = \frac{d\frac{1}{2}mv(t)^2}{dt}.$$

(Remember that m is a constant.) This says that the quantity

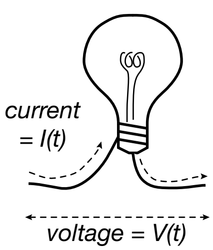
$$K(t) = \tfrac{1}{2}mv(t)^2$$

is the antiderivative we need to do the integral (72). We get

$$W = \int_{t_a}^{t_b} m\frac{dv(t)}{dt}\, v(t)\, dt = \int_{t_a}^{t_b} K'(t)\, dt = K(t_b) - K(t_a).$$

In Newtonian mechanics the quantity $K(t)$ is called ***the kinetic energy*** of the object, and our computation shows that *the amount by which the kinetic energy of an object increases is equal to the amount of work done on the object.*

12. Work done by an electric current



If at time t an electric current $I(t)$ (measured in Ampère) flows through an electric circuit, and if the voltage across this circuit is $V(t)$ (measured in Volts) then the energy supplied tot the circuit per second is $I(t)V(t)$. Therefore the total energy supplied during a time interval $t_0 \le t \le t_1$ is the integral

$$\text{Energy supplied} = \int_{t_0}^{t_1} I(t)V(t)dt.$$

(measured in Joule; the energy consumption of a circuit is defined to be how much energy it consumes per time unit, and the power consumption of a circuit which consumes 1 Joule per second is said to be one Watt.)

If a certain voltage is applied to a simple circuit (like a light bulb) then the current flowing through that circuit is determined by the resistance R of that circuit by *Ohm's law*[2] which says

$$I = \frac{V}{R}.$$

[2] http://en.wikipedia.org/wiki/Ohm's_law

So if we always choose $\delta \le 1$, then we will always have

$$|x^3 - 27| \le 37\delta \quad \text{for} \quad |x-3| < \delta.$$

Hence, if we choose $\delta = \min\left\{1, \frac{1}{37}\varepsilon\right\}$ then $|x-3| < \delta$ guarantees $|x^3 - 27| < \varepsilon$.

44 $f(x) = \sqrt{x}$, $a = 4$, $L = 2$.
You have

$$\sqrt{x} - 2 = \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}+2} = \frac{x-4}{\sqrt{x}+2}$$

and therefore

$$|f(x) - L| = \frac{1}{\sqrt{x}+2}|x-4|. \tag{73}$$

Once again it would be nice if we could replace $1/(\sqrt{x}+2)$ by a constant, and we achieve this by always choosing $\delta \le 1$. If we do that then for $|x-4| < \delta$ we always have $3 < x < 5$ and hence

$$\frac{1}{\sqrt{x}+2} < \frac{1}{\sqrt{3}+2},$$

since $1/(\sqrt{x}+2)$ increases as you decrease x.
So, if we always choose $\delta \le 1$ then $|x-4| < \delta$ guarantees

$$|f(x) - 2| < \frac{1}{\sqrt{3}+2}|x-4|,$$

which prompts us to choose $\delta = \min\left\{1, (\sqrt{3}+2)\varepsilon\right\}$.
A smarter solution: We can replace $1/(\sqrt{x}+2)$ by a constant in (73), because for all x in the domain of f we have $\sqrt{x} \ge 0$, which implies

$$\frac{1}{\sqrt{x}+2} \le \frac{1}{2}.$$

Therefore $|\sqrt{x} - 2| \le \frac{1}{2}|x-4|$, and we could choose $\delta = 2\varepsilon$.

45 Hints:

$$\sqrt{x+6} - 3 = \frac{x+6-9}{\sqrt{x+6}+3} = \frac{x-3}{\sqrt{x+6}+3}$$

so

$$|\sqrt{x+6} - 3| \le \tfrac{1}{3}|x-3|.$$

46 We have

$$\left|\frac{1+x}{4+x} - \frac{1}{2}\right| = \left|\frac{x-2}{4+x}\right|.$$

If we choose $\delta \le 1$ then $|x-2| < \delta$ implies $1 < x < 3$ so that

$$\underline{\frac{1}{7} < \text{we don't care}}\ \frac{1}{4+x} < \frac{1}{5}.$$

Therefore

$$\left|\frac{x-2}{4+x}\right| < \tfrac{1}{5}|x-2|,$$

so if we want $|f(x) - \frac{1}{2}| < \varepsilon$ then we must require $|x-2| < 5\varepsilon$. This leads us to choose

$$\delta = \min\{1, 5\varepsilon\}.$$

51 The equation (7) already contains a function f, but that is not the right function. In (7) Δx is the variable, and $g(\Delta x) = (f(x+\Delta x) - f(x))/\Delta x$ is the function; we want $\lim_{\Delta x \to 0} g(\Delta x)$.

67 $A(\frac{2}{3}, -1)$; $B(\frac{2}{5}, 1)$; $C(\frac{2}{7}, -1)$; $D(-1, 0)$; $E(-\frac{2}{5}, -1)$.

68 False! The limit must not only exist *but also be equal to* $f(a)$!

69 There are of course many examples. Here are two: $f(x) = 1/x$ and $f(x) = \sin(\pi/x)$ (see §7.3)

70 False! Here's an example: $f(x) = \frac{1}{x}$ and $g(x) = x - \frac{1}{x}$. Then f and g don't have limits at $x = 0$, but $f(x) + g(x) = x$ *does* have a limit as $x \to 0$.

71 False again, as shown by the example $f(x) = g(x) = \frac{1}{x}$.

79 $\sin 2\alpha = 2\sin\alpha\cos\alpha$ so the limit is $\lim_{\alpha\to 0}\frac{2\sin\alpha\cos\alpha}{\sin\alpha} = \lim_{\alpha\to 0} 2\cos\alpha = 2$.

Other approach: $\dfrac{\sin 2\alpha}{\sin\alpha} = \dfrac{\frac{\sin 2\alpha}{2\alpha}}{\frac{\sin\alpha}{\alpha}} \cdot \dfrac{2\alpha}{\alpha}$. Take the limit and you get 2.

80 $\frac{3}{2}$.

81 Hint: $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Answer: the limit is 1.

82 $\frac{\tan 4\alpha}{\sin 2\alpha} = \frac{\tan 4\alpha}{4\alpha} \cdot \frac{2\alpha}{\sin 2\alpha} \cdot \frac{4\alpha}{2\alpha} = 1 \cdot 1 \cdot 2 = 2$

83 Hint: multiply top and bottom with $1 + \cos x$.

84 Hint: substitute $\theta = \frac{\pi}{2} - \varphi$, and let $\varphi \to 0$. Answer: -1.

90 Substitute $\theta = x - \pi/2$ and remember that $\cos x = \cos(\theta + \frac{\pi}{2}) = -\sin\theta$. You get

$$\lim_{x\to\pi/2}\frac{x - \frac{\pi}{2}}{\cos x} = \lim_{\theta\to 0}\frac{\theta}{-\sin\theta} = -1.$$

91 Similar to the previous problem, once you use $\tan x = \frac{\sin x}{\cos x}$. The answer is again -1.

93 Substitute $\theta = x - \pi$. Then $\lim_{x\to\pi}\theta = 0$, so

$$\lim_{x\to\pi}\frac{\sin x}{x-\pi} = \lim_{\theta\to 0}\frac{\sin(\pi+\theta)}{\theta} = -\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = -1.$$

Here you have to remember from trigonometry that $\sin(\pi+\theta) = -\sin\theta$.

95 Note that the limit is for $x \to \infty$! As x goes to infinity $\sin x$ oscillates up and down between -1 and $+1$. Dividing by x then gives you a quantity which goes to zero. To give a good proof you use the Sandwich Theorem like this:

Since $-1 \le \sin x \le 1$ for all x you have

$$\frac{-1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

Since both $-1/x$ and $1/x$ go to zero as $x \to \infty$ the function in the middle must also go to zero. Hence

$$\lim_{x\to\infty}\frac{\sin x}{x} = 0.$$

97 No. As $x \to 0$ the quantity $\sin\frac{1}{x}$ oscillates between -1 and $+1$ and does not converge to any particular value. Therefore, no matter how you choose k, it will never be true that $\lim_{x\to 0}\sin\frac{1}{x} = k$, because the limit doesn't exist.

98 The function $f(x) = (\sin x)/x$ is continuous at all $x \ne 0$, so we only have to check that $\lim_{x\to 0} f(x) = f(0)$, i.e. $\lim_{x\to 0}\frac{\sin x}{2x} = A$. This only happens if you choose $A = \frac{1}{2}$.

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