From these diagrams it is clear that $A - B \neq B - A$, if $A \neq B$.

If $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8, 10\}$ then $A - B = \{1, 3, 5\}$, $B - A = \{8, 10\}$

Some properties of difference operation are as under :

(1) $\mathbf{U - A = A'}$ **(quite obvious !)**

(2) $\mathbf{A \subset B \Rightarrow A - B = \emptyset}$

$$
\begin{aligned}
A - B &= \{x \mid x \in A \text{ and } x \notin B\} \\
&= \{x \mid x \in A \text{ and } x \in B'\} \\
&= \{x \mid x \in B \text{ and } x \in B'\} \qquad (A \subset B) \\
&= \emptyset
\end{aligned}
$$

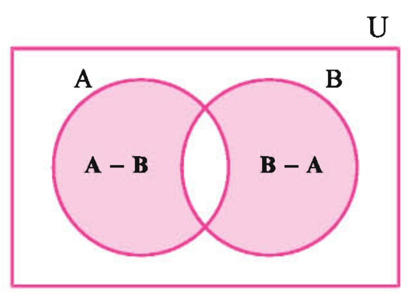
(5) Symmetric Difference set : For sets A, B $\in$ P(U), the set consisting of all elements which are in the set A or in the set B, but not in both is called the symmetric difference of the sets A and B. Symmetric difference of two sets is denoted by A Δ B.

Thus $\mathbf{A \Delta B = (A \cup B) - (A \cap B)}$

We will prove that, $A \Delta B = (A - B) \cup (B - A)$

$$
\begin{aligned}
(A \cup B) - (A \cap B) &= (A \cup B) \cap (A \cap B)' && (A - B = A \cap B') \\
&= (A \cup B) \cap (A' \cup B') && \text{(De Morgan's law)} \\
&= ((A \cup B) \cap A') \cup ((A \cup B) \cap B') && \text{(Distributive law)} \\
&= [(A \cap A') \cup (B \cap A')] \cup [(A \cap B') \cup (B \cap B')] \\
&= [\emptyset \cup (B \cap A')] \cup [(A \cap B') \cup \emptyset] \\
&= (B \cap A') \cup (A \cap B') \\
&= (B - A) \cup (A - B) \\
&= (A - B) \cup (B - A)
\end{aligned}
$$

This set is depicted below by the coloured region in the Venn diagram 2.6.



A Δ B

Figure 2.6

Just as we have ordered pairs (x, y), we can have ordered triplet or ordered n-tuple $(x_1, x_2, x_3, \ldots, x_n)$. If A, B and C are non-empty sets, their cartesian product is defined as

$A \times B \times C = \{(x, y, z) \mid x \in A, y \in B, z \in C\}$

In analogy with $A^2 = A \times A$, we will write $A^3 = A \times A \times A$.

Example 12 : Find $A \times B$, if $A = \{a, b, c\}$, $B = \{a, b\}$

Solution : $A \times B = \{(a, a), (a, b), (b, a), (b, b), (c, a), (c, b)\}$

Example 13 : If $A = \{1, 2, 3\}$, $B = \{2, 6, 7\}$, $C = \{2, 7\}$,

verify $A \times (B - C) = (A \times B) - (A \times C)$

Solution : Here $B - C = \{6\}$

$\therefore \quad A \times (B - C) = \{(1, 6), (2, 6), (3, 6)\}$

Now $A \times B = \{(1, 2), (1, 6), (1, 7), (2, 2), (2, 6), (2, 7), (3, 2), (3, 6), (3, 7)\}$

$A \times C = \{(1, 2), (1, 7), (2, 2), (2, 7), (3, 2), (3, 7)\}$

$\therefore \quad (A \times B) - (A \times C) = \{(1, 6), (2, 6), (3, 6)\}$

Thus, $A \times (B - C) = (A \times B) - (A \times C)$

Example 14 : If $A \neq \emptyset$ and $A \times B = A \times C$, show that $B = C$.

Solution : If $B = C = \emptyset$, then $A \times B = A \times C = \emptyset$ and $B = C$.

Obviously, only one of B or C being empty is not possible since $A \neq \emptyset$.

So suppose $B \neq \emptyset$, $C \neq \emptyset$.

Since $A \neq \emptyset$, there exists an $x \in A$.

$\therefore \quad$ For every $y \in B, (x, y) \in A \times B$

$\therefore \quad (x, y) \in A \times C \qquad$ **$(A \times B = A \times C)$**

$\therefore \quad x \in A, y \in C$

Hence, $\forall y, y \in B \Rightarrow y \in C$

$\therefore \quad B \subset C$

Similarly, it can be proved that $C \subset B$

$\therefore \quad B = C$

Example 15 : $A = \{ 1, 2, 3, 4 \}$, $B = \{ (a, b) \mid b$ is divisible by a; $a, b \in A \}$.

List elements of B.

Solution : Here, 1, 2, 3, 4 are divisible by 1; 2, 4 are divisible by 2; 3 is divisible by 3; and 4 is divisible by 4.

$\therefore \quad B = \{ (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4) \}$

Example 16 : If $A \times A = B \times B$, then prove that $A = B$.

Solution : If $A = \emptyset$, then $\emptyset = B \times B \Rightarrow B = \emptyset$. This shows that $A = B$.

Suppose $A \neq \emptyset$. Let $x \in A$

$\therefore \quad (x, x) \in A \times A$

(4) Given that $B \subset A$

$\therefore \quad A \cap B = B$

Now from the result (3) proved above we have

$n(A - B) = n(A) - n(A \cap B)$

$\qquad\quad = n(A) - n(B) \qquad\qquad$ **($A \cap B = B$)**

(5) $A \cap A' = \emptyset$ and $A \cup A' = U$

$\therefore \quad n(U) = n(A) + n(A')$

$\therefore \quad n(A') = n(U) - n(A)$

Note that if **$A \subset B$ then $n(A) \leq n(B)$.**

For finite sets A and B, **$n(A \times B) = n(A)\, n(B)$.**

Example 19 : If $A = \{1, 2, 3, 4\}$, $B = \{2, 4\}$, verify $n(A \times B) = n(A)\, n(B)$.

Solution : Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4\}$

Here $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4), (4, 2), (4, 4)\}$

$\therefore \quad n(A \times B) = 8$

Also $n(A) = 4$, $n(B) = 2$, $n(A \times B) = 8$

$\therefore \quad n(A \times B) = n(A)\, n(B)$

Example 20 : A and B are non singleton and $n(A \times B) = 21$. Also $A \subset B$. Find $n(A)$ and $n(B)$.

Solution : $n(A \times B) = 21 = 3 \times 7 = 1 \times 21$

But, $n(A) \neq 1$, $n(B) \neq 1$

$\therefore \quad n(A) = 3$ and $n(B) = 7$ or $n(A) = 7$ and $n(B) = 3$.

But, $n(A) \leq n(B) \qquad\qquad$ **($A \subset B$)**

$\therefore \quad n(A) = 3,\ n(B) = 7$

Example 21 : In a troop of 20 dancers performing Bharatnatyam or Kuchipudi, 12 dancers perform Bharatnatyam and 4 perform both Bharatnatyam and Kuchipudi. Find the number of dancers performing Kuchipudi.

(**Note :** Each dancer performs Bharatnatyam or Kuchipudi)

Solution : Let A = Set of dancers performing Bharatnatyam

B = Set of dancers performing Kuchipudi

Then given that, $n(A) = 12$, $n(A \cap B) = 4$, $n(A \cup B) = 20$

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\therefore \quad 20 = 12 + n(B) - 4$

$\quad\quad 20 = n(B) + 8$

$\therefore \quad n(B) = 12$

Thus number of dancers performing Kuchipudi is 12.

$\therefore$ Number of Dancers performing only Kuchipudi $= n(B) - n(A \cap B)$

$= 12 - 4 = 8$

Example 22 : In a group of people, 28 like Gujarati movies, 30 like Hindi movies, 42 like English movies; 5 like both Gujarati and Hindi movies, 8 like Hindi and English movies, 8 like Gujarati and English movies and 3 like Gujarati, Hindi and English movies. What is the least number of people in the group ?

Solution :

Here let G = the set of people who like Gujarati movies

H = the set of people who like Hindi movies

E = the set of people who like English movies

then given that

$n(G) = 28$, $n(H) = 30$, $n(E) = 42$

$n(G \cap H) = 5$, $n(E \cap H) = 8$, $n(G \cap E) = 8$, $n(G \cap E \cap H) = 3$

Now, $n(G \cup E \cup H) = n(G) + n(H) + n(E) - n(G \cap H) - n(E \cap H) - n(G \cap E) + n(G \cap E \cap H)$

$= 28 + 30 + 42 - 5 - 8 - 8 + 3$

$= 103 - 21 = 82$

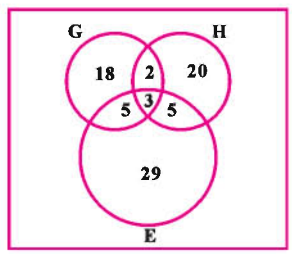


Figure 2.9

Some persons may not like to watch a movie. Hence there are at least 82 people in the group.

Miscellaneous Problems

Example 23 : If $A \cap B = A \cap C$, $A \cup B = A \cup C$, then prove that $B = C$ $(B \neq \emptyset, C \neq \emptyset)$

Method 1 : Let $x \in B$

$\therefore \quad x \in A \cup B$

$\therefore \quad x \in A \cup C$ $\quad\quad$ $(A \cup B = A \cup C)$

there are two possibilities

(1) $x \in A$ or (2) $x \in C$

(1) $x \in A$

$\therefore \quad B \not\subset C$ is not true.

$\therefore \quad B \subset C$.

Similarly $C \subset B$.

Hence $B = C$.

Example 25 : Prove that $P(A) = P(B) \Rightarrow A = B$

Solution : $A \subset A \Rightarrow A \in P(A)$

$\Rightarrow A \in P(B)$ **(P(A) = P(B))**

$\Rightarrow A \subset B$

Similarly $B \subset A$.

$\therefore \quad A = B$

Example 26 : $n(A \times A) = 9$. $(a, b) \in A \times A$. Also $c \in A$. Write the set A.

Solution : Let $n(A) = k$

Now, $n(A \times A) = k^2 = 9$

$\therefore \quad k = 3$

$(a, b) \in A \times A$

$\therefore \quad a \in A,\ b \in A$

Further we are given that $c \in A$.

Thus A has 3 elements namely a, b, c.

$\therefore \quad A = \{a, b, c\}$

Example 27 : $A \cap B = \emptyset$ and $A \cup B = U$, prove that $A' = B$.

Solution : Let $x \in B$

$\therefore \quad x \notin A$ as $A \cap B = \emptyset$

$\therefore \quad x \in A'$

$\therefore \quad B \subset A'$ **(i)**

Let $x \in A'$

$\therefore \quad x \notin A$

but $x \in U$

$\therefore \quad x \in A \cup B$ **(A ∪ B = U)**

$\therefore \quad x \in A$ or $x \in B$

$\therefore \quad x \in B$ **(x ∉ A)**

$\therefore \quad A' \subset B$ **(ii)**

From (i) and (ii) $A' = B$.