Trigonometry is very widely applied in Astronomy, Physics, Engineering and various branches of mathematics. The word 'trigonometry' is derived from two Greek words “trigono” and “metron”. The word “trigono” means a “triangle” and the word “metron” mean “to measure”. Hence the word trigonometry means ‘measurement of a triangle’. In recent years, its application has been extended beyond the measurement of triangles. A class of functions called trigonometric functions forms the basis of the study of periodic phenomena like mechanical vibrations, motions of waves and so on. We obtained preliminary introduction to trigonometry in standard X. At that time we studied trigonometrical ratios like *sin*, *cos*, *tan* etc. for acute angles. This study was confined only to acute angles of a right angled triangle. Now we shall study trigonometric functions in a wider sense.

4.2 Trigonometric Point

In the coordinate plane, the circle whose centre is at the origin and whose radius is one unit is called the unit circle.

The unit circle intersects X-axis at A(1, 0) and A'(−1, 0) and Y-axis at B(0, 1) and B'(0, −1) as radius is one unit.

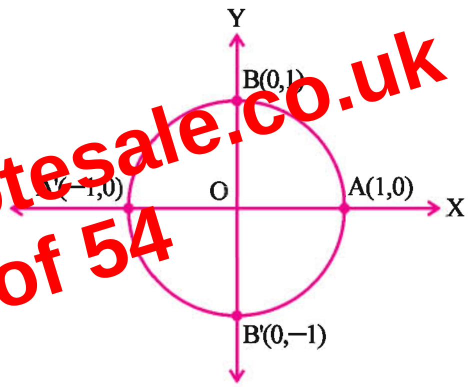

Figure 4.1

Let θ be any real number. Then we can obtain a unique point on the unit circle corresponding to this given real number θ.

If $\theta = 0$, we take the point corresponding to θ as A(1, 0). If $0 < \theta < 2\pi$, then a unique point P exists on the unit circle such that the length of $\overset{\frown}{AP}$ is θ. We measure the arc from A to P in anticlockwise direction. Again we assume the continuity of the arc and hence since circumference of unit circle is 2π, for every $\theta \in (0, 2\pi)$ there is an arc of length θ such that $l(\overset{\frown}{AP}) = \theta$. The point P thus obtained is called a trigonometric point. This point P is the point corresponding to $\theta \in [0, 2\pi)$ and is denoted by P(θ).

Now let $\theta \in$ R.

Let $\left[\frac{\theta}{2\pi}\right] = n$

Then n is an integer and

$$n \le \frac{\theta}{2\pi} < n + 1$$

In general $\theta = 2n\pi + \alpha$, $n \in$ Z.

If $\alpha = 0$, then $\theta = 2n\pi$ and if $\alpha = \pi$, $\theta = 2n\pi + \pi$, $n \in$ Z.

$\therefore\quad \theta = 2n\pi$ or $\theta = (2n + 1)\pi$, $n \in$ Z

As $2n\pi$ is an even multiple of π and $(2n + 1)\pi$ is an odd multiple of π, we see that $sin\theta = 0 \Rightarrow \theta$ is an integral multiple of π. i.e. $\theta = k\pi$, $k \in$ Z.

Conversely, if $\theta = k\pi$, $k \in$ Z, then P(θ) is A or A' and hence $sin\theta = 0$

Thus the **set of zeroes of *sin* is {$k\pi$ | $k \in$ Z}.**

Zeroes of *cosine* : Suppose for some $\theta \in$ R *cosine* function has value zero, that is $cos\theta = 0$.

$\therefore\quad$ T-point P(θ) has x-coordinate 0.

$\therefore\quad$ P(θ) is on Y-axis.

$\therefore\quad$ P(θ) = B(0, 1) or B'(0, -1)

We know that B and B' correspond to $\alpha = \frac{\pi}{2}$ and $\alpha = \frac{3\pi}{2}$.

In general $\theta = 2n\pi + \alpha$, $n \in$ Z.

$\therefore\quad \theta = 2n\pi + \frac{\pi}{2}$ or $\theta = 2n\pi + \frac{3\pi}{2}$, $n \in$ Z

Thus $\theta = (4n + 1)\frac{\pi}{2}$ or $\theta = (4n + 3)\frac{\pi}{2}$, $n \in$ Z

$4n + 1 = 2(2n) + 1$, $4n + 3 = 2(2n + 1) + 1$

$\therefore\quad 4n + 1$ and $4n + 3$ are of form $2k + 1$, $k \in$ Z

$\therefore\quad (4n + 1)$ or $(4n + 3)$, $n \in$ Z is a form of odd integers, so we see that, θ is an odd multiple of $\frac{\pi}{2}$.

$\therefore\quad \theta = (2k - 1)\frac{\pi}{2}$ or $\theta = (2k + 1)\frac{\pi}{2}$, $k \in$ Z

Conversely, it is clear that if $\theta = (2k + 1)\frac{\pi}{2}$, $k \in$ Z.

then $\theta = (2(2n) + 1)\frac{\pi}{2}$ or $\theta = (2(2n + 1) + 1)\frac{\pi}{2}$ **(according as *k* is even or odd.)**

$\therefore\quad \theta = (4n + 1)\frac{\pi}{2}$ or $(4n + 3)\frac{\pi}{2}$

$\therefore\quad \theta = 2n\pi + \frac{\pi}{2}$ or $\theta = 2n\pi + \frac{3\pi}{2}$

$\therefore\quad \alpha = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\therefore\quad$ P(θ) = P(α) = B or B'

$\therefore\quad$ x-coordinate of P(θ) is zero.

$\therefore\quad cos\theta = 0$

4.12 Measures of Angles

We will learn about two methods of measuring an angle in this section.

Degree Measure : In this system a right angle is divided into ninety congruent parts. Each part is said to have measure one degree, written as 1^{o}. Thus, one degree is one-ninetieth part of the measure of a right angle. A degree is further divided in 60 equal parts and each part is called a minute. The symbol 1' is used to denote one minute. One minute is further divided in 60 equal parts, each part is called a second. The symbol 1" is used to denote one second.

Thus, $1^{o} = 60' = 60$ minutes.

$1' = 60'' = 60$ seconds.

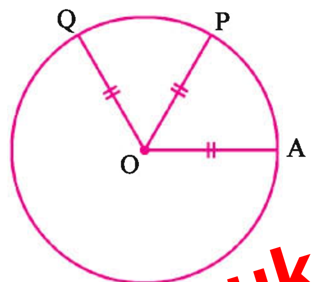

Figure 4.11

Radian Measure : Radian measure is another unit for measurement of an angle. One radian is the measure of the angle subtended at the centre of a circle by an arc of length equal to the radius of the circle. It is denoted by 1^{c}. (c for circular measure.) It is also denoted by 1^{R}.

Consider a circle of radius r having centre O. Let A be a point on the circle. Now cut off an arc AP whose length is equal to the radius r of the circle. i.e. $l(\overset{\frown}{AP}) = r$. Then the measure of ∠AOP is 1 radian ($= 1^{c}$). If $l(\overset{\frown}{AQ}) = 2r$, then the measure of ∠AOQ is 2 radian ($= 2^{c}$). Since a radian is the unit of measurement of an angle, it should be a constant quantity not depending upon the radius of the circle.

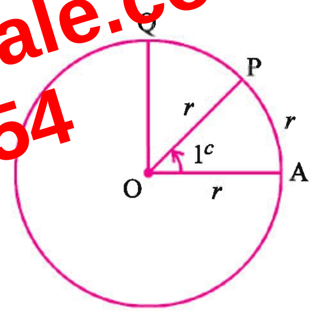

Figure 4.12

Consider a circle with centre O and radius r. Take a point A on the circle and cut off an $\overset{\frown}{AP}$ whose length is equal to the radius r. Draw $\overline{OA}$ and $\overline{OP}$ and draw $\overline{OQ} \perp \overline{OA}$. Now by definition, $m\angle AOP = 1^{c}$ and ∠AOQ is a right angle.

Since in a circle, the angles at the centre of a circle have measures proportional to the lengths of arcs subtending them,

$$\frac{m\angle AOP}{m\angle AOQ} = \frac{l(\overset{\frown}{AP})}{l(\overset{\frown}{AQ})}$$

$$\therefore \quad \frac{m\angle AOP}{m\angle AOQ} = \frac{r}{\frac{1}{2}(\pi r)} \qquad \left(l(\overset{\frown}{AQ}) = \frac{1}{4}(2\pi r) = \frac{1}{2}\pi r\right)$$

$$\therefore \quad \frac{1^{c}}{m\angle AOQ} = \frac{2}{\pi}$$

$$\therefore \quad m\angle AOQ = \frac{\pi}{2} \text{ radian}$$

∴ Radian measure of a right angle is $\frac{\pi}{2}$.

∴ Radian is a constant angle and Radian measure of a right angle $= \frac{\pi}{2}$

Since degree measure of an angle lies in (0, 180) and $0^o = 0^c$ and $180^o = \pi^c$, the radian measure of an angle lies in $(0, \pi)$.

Example 12 : Convert $47^o\ 30'$ into radian measure.

Solution : We know that $1^o = 60'$

$$\therefore \quad 30' = \left(\frac{30}{60}\right)^o = \left(\frac{1}{2}\right)^o$$

$$47^o\ 30' = \left(47\frac{1}{2}\right)^o = \left(\frac{95}{2}\right)^o$$

We know that $180^o = \pi^c$

$$\therefore \quad \left(\frac{95}{2}\right)^o = \left(\frac{\pi}{180} \times \frac{95}{2}\right)^c = \left(\frac{19\pi}{72}\right)^c = \frac{19\pi}{72}$$

Hence, radian measure of the angle with degree measure $47^o\ 30'$ is $\left(\frac{19\pi}{72}\right)^c$ or $\frac{19\pi}{72}$.

Example 13 : Convert $39^o\ 22'\ 30''$ into radian measure.

Solution : $60'' = \left(\frac{30}{60}\right)' = \left(\frac{1}{2}\right)'$

$$\therefore \quad 22'\ 30'' = \left(22\frac{1}{2}\right)' = \left(\frac{45}{2}\right)'$$

$$\left(\frac{45}{2}\right)' = \left(\frac{45}{2} \times \frac{1}{60}\right)^o = \left(\frac{3}{8}\right)^o$$

$$\therefore \quad 39^o\ 22'\ 30'' = \left(39\frac{3}{8}\right)^o = \left(\frac{315}{8}\right)^o$$

$$\left(\frac{315}{8}\right)^o = \left(\frac{315}{8} \times \frac{\pi}{180}\right)^c = \left(\frac{7\pi}{32}\right)^c$$

Hence, radian measure of the angle with degree measure $39^o\ 22'\ 30''$ is $\left(\frac{7\pi}{32}\right)^c = \frac{7\pi}{32}$.

Example 14 : Convert 2 radian into degree measure.

Solution : We know that $\pi^c = 180^o$

$$\therefore \quad 2^c = \left(\frac{180}{\pi} \times 2\right)^o = \left(\frac{180 \times 7 \times 2}{22}\right)^o$$

$$= \left(114\frac{6}{11}\right)^o = 114^o + \left(\frac{6}{11} \times 60\right)' \qquad \text{(as } 1^o = 60'\text{)}$$

$$= 114^o + \left(32\frac{8}{11}\right)' = 114^o + 32' + \left(\frac{8}{11} \times 60\right)'' \quad \text{(as } 1' = 60''\text{)}$$

$$= 114^o + 32' + 44''$$

Hence, 2 radian $= 114^o\ 32'\ 44''$

Example 15 : Find in degree measure, the measure of the angle subtended at the centre of a circle of radius 25 *cm* by an arc of length 55 *cm*.

Solution : Here, $r = 25\ cm$, $l = 55\ cm$

We have, $\theta = \left(\frac{l}{r}\right)^c$

$$\therefore \quad \theta = \left(\frac{55}{25}\right)^c = \left(\frac{11}{5} \times \frac{180}{\pi}\right)^o = \left(\frac{11 \times 36 \times 7}{22}\right)$$

$$\therefore \quad \theta = 126^o$$

If $\frac{\pi}{2} < \theta < \pi$, then $P(\theta) = P(x, y)$ is in the 2nd quadrant. In the second quadrant, $x < 0$ and $y > 0$. So, $x = cos\theta < 0$, $y = sin\theta > 0$.

If $\pi < \theta < \frac{3\pi}{2}$, then $P(\theta) = P(x, y)$ is in the 3rd quadrant and in the third quadrant $x < 0$, $y < 0$. So, $x = cos\theta < 0$ and $y = sin\theta < 0$.

If $\frac{3\pi}{2} < \theta < 2\pi$, then $P(\theta) = P(x, y)$ is in the 4th quadrant. In the 4th quadrant $x > 0$, $y < 0$. So, $x = \cos\theta > 0$ and $y = sin\theta < 0$.

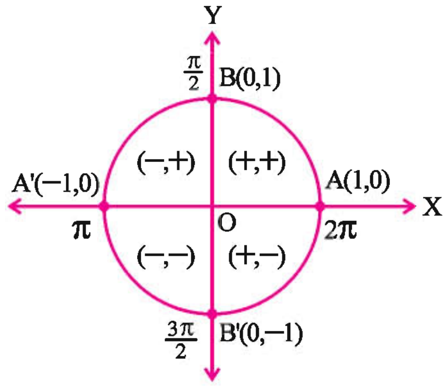

Figure 4.19

As $tan\theta = \frac{sin\theta}{cos\theta}$, $cot\theta = \frac{cos\theta}{sin\theta}$, $cosec\theta = \frac{1}{sin\theta}$ and $sec\theta = \frac{1}{cos\theta}$, we can find the signs of other trigonometric functions in different quadrants.

Function \ Quadrant	First	Second	Third	Fourth
sin	+	+	−	−
cos	+	−	−	+
tan	+	−	+	−
cot	+	−	+	−
cosec	+	+	−	−
sec	+	−	−	+

Example 21 : If $cot\theta = \frac{-5}{12}$, and $P(\theta)$ lies in the second quadrant, find the value of other trigonometric functions.

Solution : Since $cot\theta = \frac{-5}{12}$, we have $tan\theta = \frac{-12}{5}$

Now, $sec^2\theta = 1 + tan^2\theta$

$$= 1 + \frac{144}{25} = \frac{169}{25}$$

$\therefore \quad sec\theta = \pm\frac{13}{5}$

Since $P(\theta)$ is in the second quadrant, $sec\theta$ will be negative.

$\therefore \quad sec\theta = -\frac{13}{5}$ and $cos\theta = \frac{-5}{13}$

(15) The value of the expression $sin^6\theta + cos^6\theta + 3sin^2\theta\ cos^2\theta$ is ......

(a) 0 (b) 1 (c) 2 (d) greater than 3

(16) The expression $tan^2\alpha + cot^2\alpha$ is...

(a) ≥ -2 (b) ≥ 2 (c) ≤ 2 (d) ≤ -2

(17) If $cosec\theta + cot\theta = \frac{5}{2}$, then the value of $tan\theta$

(a) $\frac{14}{24}$ (b) $\frac{20}{21}$ (c) $\frac{21}{20}$ (d) $\frac{15}{16}$

(18) $1 - \frac{sin^2\theta}{1 + cos\theta} + \frac{1 + cos\theta}{sin\theta} - \frac{sin\theta}{1 - cos\theta}$ equals...

(a) 0 (b) 1 (c) $sin\theta$ (d) $cos\theta$

(19) If $sec\theta = \sqrt{2}$, $\frac{3\pi}{2} < \theta < 2\pi$, then $\frac{1 + tan\theta + cosec\theta}{1 + cot\theta - cosec\theta}$ is...

(a) $-\sqrt{2}$ (b) -1 (c) $\frac{1}{\sqrt{2}}$ (d) 0

(20) If $p = a\ cos^2\theta\ sin\theta$ and $q = a\ sin^2\theta\ cos\theta$, then $\frac{(p^2 + q^2)^3}{p^2q^2}$ is...

(a) $\frac{1}{a}$ (b) a^2 (c) a (d) a^3

(21) If $secA + tanA = \frac{a+1}{a-1}$, then $cosA$ is...

(a) $\frac{2a}{a^2 - 1}$ (b) $\frac{2a}{a^2 + 1}$ (c) $\frac{a^2 + 1}{a^2 - 1}$ (d) $\frac{a^2 - 1}{a^2 + 1}$

Summary

1. Trigonometric point, Trigonometric point function, Period
2. *sine* function, *cosine* function, their zeroes and range, fundamental identity
3. Other trigonometric functions, their ranges, identities
4. Increasing and decreasing functions
5. Degree measure and radian measure
6. Even and odd functions
7. Right angled triangle and related trigonometric functions
8. Values of trigonometric functions in each quadrant.

Length of minor $\widehat{AP}$ is $\frac{\pi}{3}$.

$\therefore \quad m\angle AOP = \frac{\pi}{3} = 60^{\circ}$

In ΔOAP, $OA = OP$ **(Radii)**

$\therefore \quad m\angle OPA = m\angle OAP$ **(i)**

As $m\angle AOP = 60^{\circ}$,

$m\angle OPA + m\angle OAP = 120^{\circ}$

$\therefore$ From (i), $m\angle OPA = m\angle OAP = 60^{\circ}$

$\therefore \quad \Delta OAP$ is an equilateral triangle.

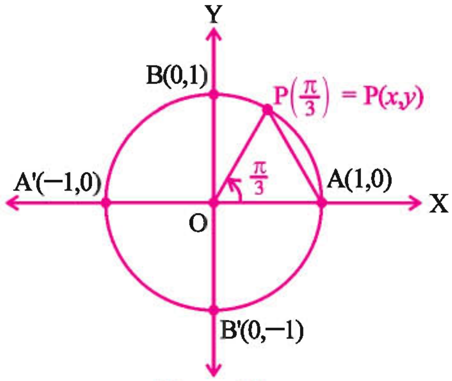

Figure 5.3

Again $OA = OP = 1$ **(Radii of unit circle)**

$\therefore \quad AP = 1$

$\therefore \quad AP^2 = 1$

Now, since $P(x, y)$ and $A(1, 0)$,

$\therefore \quad (x-1)^2 + (y-0)^2 = 1$

$\therefore \quad x^2 - 2x + 1 + y^2 = 1$

but $\quad x^2 + y^2 = 1$

$\therefore \quad 2x = 1$

$\therefore \quad x = \frac{1}{2}$

Again $x^2 + y^2 = 1$

$\therefore \quad \frac{1}{4} + y^2 = 1$

$\therefore \quad y^2 = \frac{3}{4}$

$\therefore \quad y = \frac{\sqrt{3}}{2}$ **$\left(P\left(\frac{\pi}{3}\right)\right.$ is in the first quadrant, $\left. y > 0\right)$**

$\therefore$ Coordinates of $P\left(\frac{\pi}{3}\right)$ are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

$\therefore \quad \cos\frac{\pi}{3} = \frac{1}{2}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. So, $\tan\frac{\pi}{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$.

$\therefore \quad \sec\frac{\pi}{3} = 2$, $\operatorname{cosec}\frac{\pi}{3} = \frac{2}{\sqrt{3}}$, $\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$.

5.5 Coordinates of $P\left(\frac{\pi}{6}\right)$:

Suppose the coordinates of $P\left(\frac{\pi}{6}\right)$ are (x, y).

Length of minor $\widehat{AP}$ is $\frac{\pi}{6}$.

If $180 < \theta < 360$, then $-180 < \theta - 360 < 0$.

$\therefore \quad 0 < 360 - \theta < 180$

Hence, $\overrightarrow{OQ}$ rotates in the half-plane below X-axis in clockwise direction and without passing through A again takes the position of $\overrightarrow{OP}$. We get $m\angle AOP = 360 - \theta$ and we get angle of general measure θ as shown in 5.21. Thus if $\theta = 210$, $360 - \theta = 360 - 210 = 150$. $\angle AOP$ with general measure 210^o is shown in figure 5.21.

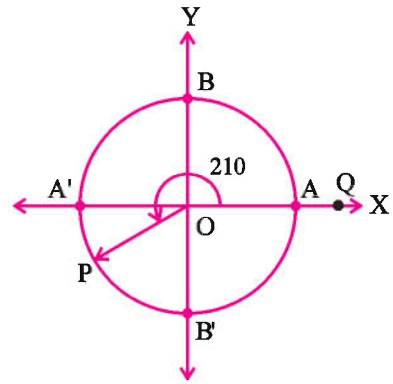

Figure 5.21

If $\theta \notin [0, 360)$ and $\theta > 0$, we can write $\theta = 360n + \alpha$, where $n = \left[\frac{\theta}{360}\right]$. $n \in$ N and $0 \le \alpha < 360$. We can get angle with general measure α as described earlier. n is the number of rotations of $\overrightarrow{OQ}$ before $\overrightarrow{OQ}$ coincides with $\overrightarrow{OP}$ and this rotation is anti-clockwise and $n > 0$.

Example 11 : For $\theta = 760$, describe the angle with general measure θ^o.

Solution : $\left[\frac{\theta}{360}\right] = \left[\frac{760}{360}\right] = 2$ and $760 = 360 \cdot 2 + 40$

$\therefore \quad \alpha = 40$

Thus $\overrightarrow{OQ}$ must complete 2 rotations anti-clockwise and then we get $\overrightarrow{OP}$ in upper half plane of $\overleftrightarrow{OA}$, so that $m\angle AOP = 40$. The angle $\angle AOP$ thus, generated by rotation of $\overrightarrow{OQ}$ has general measure 760^o.

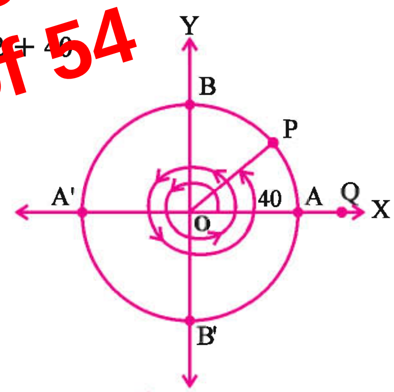

Figure 5.22

Suppose $\theta < 0$. If $-180 < \theta < 0$, then we can find a point on the unit circle below $\overleftrightarrow{AA'}$ such that $m\angle AOP = |\theta| = -\theta$ as $0 < -\theta < 180$.

If $\overrightarrow{OQ}$ rotates clockwise and without passing through A again takes position of $\overrightarrow{OP}$, we get $\angle AOP$ as angle having general measure θ^o.

So if $\theta = -60$, then we shall have P in the lower half of the unit circle such that $m\angle AOP = 60$. Thus, $\overrightarrow{OQ}$ after rotating clockwise takes position such that $\angle AOP$ has general degree measure -60.

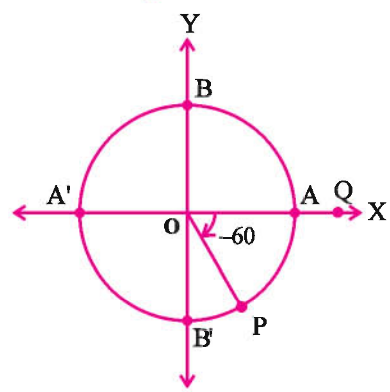

Figure 5.23

EXERCISE 5.3

1. For the following find the complete number of rotations n and α.

(1) 750° (2) 1125° (3) 1485°

2. Obtain the number of rotations n, α and then draw such an angle having general measure.

(1) 840° (2) -765° (3) -1470°

EXERCISE 5

1. Plot the graph of $y = \sin x$ and $y = \cos x$ on the same set of coordinate axes.

2. Draw the graph of $y = 3\sin 2x$.

3. Draw the graph of $y = 2\cos 3x$.

4. Obtain the number of rotations n, α and then draw such an angle having general measure.

(1) -1320° (2) -2000° (3) -54[illegible]$^\circ$

5. Select proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) Value of $\tan\left(\frac{19\pi}{3}\right)$ is... ☐

(a) $\sqrt{3}$ (b) $-\sqrt{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{-1}{\sqrt{3}}$

(2) Value of $\cot\left(\frac{-15\pi}{4}\right)$ is... ☐

(a) 1 (b) -1 (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{-1}{\sqrt{3}}$

(3) If $\sec\theta + \tan\theta = \sqrt{3}$, $0 < \theta < \pi$, then θ is equal to... ☐

(a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{-\pi}{3}$

(4) If $\tan\theta = -\frac{1}{\sqrt{5}}$ and $P(\theta)$ lies in the 4th quadrant, then the value of $\cos\theta$ is ... ☐

(a) $\frac{\sqrt{5}}{\sqrt{6}}$ (b) $\frac{2}{\sqrt{6}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{6}}$

(5) If $x \cdot \sin 45^\circ \cos^2 60^\circ = \frac{\tan^2 60^\circ \operatorname{cosec}^2 30^\circ}{\sec 45^\circ \cot^2 30^\circ}$, then $x =$ .... ☐

(a) 16 (b) 1 (c) $8\sqrt{2}$ (d) $\frac{16}{3}$