Putting $\alpha = \frac{\pi}{2}$ and $\beta = \theta$ in (iv) and (ii) respectively, we get

$$sin\left(\frac{\pi}{2}+\theta\right) = sin\frac{\pi}{2}\,cos\theta + cos\frac{\pi}{2}\,sin\theta = 1 \cdot cos\theta + 0 \cdot sin\theta = cos\theta$$

$$\therefore \quad sin\left(\frac{\pi}{2}+\theta\right) = cos\theta$$

$$cos\left(\frac{\pi}{2}+\theta\right) = cos\frac{\pi}{2}\,cos\theta - sin\frac{\pi}{2}\,sin\theta = 0 \cdot cos\theta - 1 \cdot sin\theta = -sin\theta$$

$$\therefore \quad cos\left(\frac{\pi}{2}+\theta\right) = -sin\theta$$

and hence, $tan\left(\frac{\pi}{2}+\theta\right) = -cot\theta$

Similarly putting, $\alpha = \frac{3\pi}{2}$ and $\beta = \theta$ in (i) to (iv), we get

$$sin\left(\frac{3\pi}{2}-\theta\right) = -cos\theta,\ cos\left(\frac{3\pi}{2}-\theta\right) = -sin\theta$$

$$\therefore \quad tan\left(\frac{3\pi}{2}-\theta\right) = cot\theta$$

Similarly, $sin\left(\frac{3\pi}{2}+\theta\right) = -cos\theta,\ cos\left(\frac{3\pi}{2}+\theta\right) = sin\theta$

$$\therefore \quad tan\left(\frac{3\pi}{2}+\theta\right) = -cot\theta$$

Again putting $\alpha = \pi$, $\beta = \theta$ and $\alpha = 2\pi$, $\beta = \theta$ in (i) to (iv), we can prove the following :

$sin(\pi - \theta) = sin\theta,\ cos(\pi - \theta) = -cos\theta,\ tan(\pi - \theta) = -tan\theta$

$sin(\pi + \theta) = -sin\theta,\ cos(\pi + \theta) = -cos\theta,\ tan(\pi + \theta) = tan\theta$

$sin(2\pi - \theta) = -sin\theta,\ cos(2\pi - \theta) = cos\theta,\ tan(2\pi - \theta) = -tan\theta$

$sin(2\pi + \theta) = sin\theta,\ cos(2\pi + \theta) = cos\theta,\ tan(2\pi + \theta) = tan\theta$

We will be using these formulae frequently for solving examples, so it would be very useful to remember them. As an aid to memory, remember the following.

First of all, it is enough to consider values of trigonometric functions $sin\alpha$, $cos\alpha$ etc. where $0 \le \alpha < 2\pi$, because if $\theta \in R$ then $\theta = 2n\pi + \alpha$, $0 \le \alpha < 2\pi$. We let $0 < \beta < \frac{\pi}{2}$. Then typical real numbers $\frac{\pi}{2} - \beta$, $\frac{\pi}{2} + \beta$, $\frac{3\pi}{2} - \beta$ and $\frac{3\pi}{2} + \beta$ correspond to the trigonometric points which lie in the I, II, III, IV quadrants respectively.

$\frac{\pi}{2} + \beta$	$\frac{\pi}{2} - \beta$
$\frac{3\pi}{2} - \beta$	$\frac{3\pi}{2} + \beta$

Figure 4.2

From figure 4.2 for any real value, trigonometric function change as under, *sin* $\rightarrow$ *cos, cos* $\rightarrow$ *sin, tan* $\rightarrow$ *cot, cot* $\rightarrow$ *tan, sec* $\rightarrow$ *cosec, cosec* $\rightarrow$ *sec.*

$P\left(\frac{\pi}{2}+\beta\right)$ is in second quadrant.

In the second quadrant $sin\left(\frac{\pi}{2}+\beta\right) > 0$.

Note : Choice of sign is according to the original function on the left.

$$\therefore \quad sin\left(\frac{\pi}{2}+\beta\right) = cos\beta$$

$P\left(\frac{3\pi}{2}-\beta\right)$ is in the third quadrant and in the third quadrant $cos\left(\frac{3\pi}{2}-\beta\right)$ is $-ve$.

Also, $sin\frac{5\pi}{12} = sin\left(\frac{\pi}{2}-\frac{\pi}{12}\right) = cos\frac{\pi}{12} = \frac{\sqrt{6}+\sqrt{2}}{4}$

$$cos\frac{5\pi}{12} = cos\left(\frac{\pi}{2}-\frac{\pi}{12}\right) = sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

(2) (i) $sin(\alpha+\beta)\cdot sin(\alpha-\beta) = sin^2\alpha - sin^2\beta = cos^2\beta - cos^2\alpha$

(ii) $cos(\alpha+\beta)\cdot cos(\alpha-\beta) = cos^2\alpha - sin^2\beta = cos^2\beta - sin^2\alpha$

(i) $sin(\alpha+\beta)\cdot sin(\alpha-\beta) = (sin\alpha\ cos\beta + cos\alpha\ sin\beta)(sin\alpha\ cos\beta - cos\alpha\ sin\beta)$

$$= sin^2\alpha\cdot cos^2\beta - cos^2\alpha\cdot sin^2\beta$$

$$= sin^2\alpha\ (1 - sin^2\beta) - (1 - sin^2\alpha)\cdot sin^2\beta$$

$$= sin^2\alpha - sin^2\alpha\ sin^2\beta - sin^2\beta + sin^2\alpha\ sin^2\beta$$

$$= sin^2\alpha - sin^2\beta$$

$\therefore\ sin(\alpha+\beta)\cdot sin(\alpha-\beta) = sin^2\alpha - sin^2\beta$

Now, $sin(\alpha+\beta)\cdot sin(\alpha-\beta) = sin^2\alpha - sin^2\beta$

$$= (1 - cos^2\alpha) - (1 - cos^2\beta)$$

$$= cos^2\beta - cos^2\alpha$$

$\therefore\ sin(\alpha+\beta)\cdot sin(\alpha-\beta) = cos^2\beta - cos^2\alpha$

Similarly, it can be proved that

$$cos(\alpha+\beta)\cdot cos(\alpha-\beta) = cos^2\alpha - sin^2\beta = cos^2\beta - sin^2\alpha$$

4.5 The Range of $f(\alpha) = a cos\alpha + b sin\alpha$, $\alpha \in R$, $a, b \in R$, $a^2 + b^2 \neq 0$

As $a^2 + b^2 \neq 0$, we consider three cases :

(1) $a = 0, b \neq 0$ (2) $a \neq 0, b = 0$ (3) $a \neq 0, b \neq 0$

Case (1) : $a = 0,\ b \neq 0$

Then, $f(\alpha) = bsin\alpha$. Range of $sin\alpha$ is $[-1, 1]$.

$-1 \leq sin\alpha \leq 1$

$\Leftrightarrow\ -b \leq bsin\alpha \leq b\quad (b > 0)$

$\therefore$ For $b > 0$, the range of $bsin\alpha$ is $[-b, b] = [-|b|, |b|]$. $(|b| = b)$

Now; for $b < 0$, $-1 \leq sin\alpha \leq 1 \Leftrightarrow -b \geq bsin\alpha \geq b$

$$\Leftrightarrow b \leq bsin\alpha \leq -b$$

$\therefore$ For $b < 0$, the range is $[b, -b] = [-|b|, |b|]$. $(|b| = -b)$

$\therefore$ The range of $f(\alpha) = bsin\alpha$ is $[-|b|, |b|]$.

Case (2) : $a \neq 0,\ b = 0$

Then, $f(\alpha) = acos\alpha$. Its range is $[-|a|, |a|]$ as before.

Case (3) : $a \neq 0,\ b \neq 0$

In this case, we shall express $acos\alpha + bsin\alpha$ in the form $r\,cos(\theta - \alpha)$.

As $r\,cos(\theta - \alpha) = r\,cos\theta\ cos\alpha + r\,sin\theta\ sin\alpha$, we shall find r and θ such that $a = r\,cos\theta$, $b = r\,sin\theta$. $(r > 0)$

Example 7 : Express $\sqrt{3}\sin\alpha - \cos\alpha$ in the form $r\sin(\alpha - \theta)$ and find r and θ, where, $r > 0$, $0 \le \theta < 2\pi$.

Solution : Let $f(\alpha) = \sqrt{3}\sin\alpha - \cos\alpha$

Multiplying and dividing by $\sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$,

$$f(\alpha) = 2\left(\frac{\sqrt{3}}{2}\sin\alpha - \frac{1}{2}\cos\alpha\right)$$

$$= 2\left(\sin\alpha\cos\frac{\pi}{6} - \cos\alpha\sin\frac{\pi}{6}\right)$$

$$= 2\sin\left(\alpha - \frac{\pi}{6}\right)$$

$$= r\sin(\alpha - \theta)$$

$r = 2$, $\theta = \frac{\pi}{6}$. Here $\theta = \frac{\pi}{6}$ satisfies $0 \le \theta < 2\pi$.

Example 8 : If $\sqrt{3}\cos\alpha - \sin\alpha = r\cos(\alpha - \theta)$, find r and θ. $r > 0$,

where (i) $0 < \theta < 2\pi$ (ii) $\frac{-\pi}{2} < \theta < 0$

Solution : Let $f(\alpha) = \sqrt{3}\cos\alpha - \sin\alpha$

Multiplying and dividing by $r = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$,

$$f(\alpha) = 2\left(\frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha\right)$$

$$= 2\left(\cos\frac{\pi}{6}\cos\alpha - \sin\frac{\pi}{6}\sin\alpha\right)$$

$$= 2\cos\left(\alpha + \frac{\pi}{6}\right)$$

$$= 2\cos\left(\alpha - \left(-\frac{\pi}{6}\right)\right)$$

Now comparing with $r\cos(\alpha - \theta)$, we get

$r = 2$, $\theta = \frac{-\pi}{6}$ and $\theta = \frac{-\pi}{6}$ satisfies $\frac{-\pi}{2} < \theta < 0$

$$2\cos\left(\alpha + \frac{\pi}{6}\right) = 2\cos\left(\alpha + \frac{\pi}{6} - 2\pi\right) = 2\cos\left(\alpha - \frac{11\pi}{6}\right)$$

$\therefore$ $\theta = \frac{11\pi}{6}$ satisfies $0 < \theta < 2\pi$.

Example 9 : Prove that $\sin^2 A = \cos^2(A - B) + \cos^2 B - 2\cos(A - B)\cos A\cos B$.

Solution : R.H.S. $= \cos^2(A - B) + \cos^2 B - 2\cos(A - B)\cos A\cos B$.

$$= \cos^2 B + \cos^2(A - B) - 2\cos(A - B)\cos A\cos B$$

$$= \cos^2 B + \cos(A - B)[\cos(A - B) - 2\cos A\cos B]$$

$$= \cos^2 B + \cos(A - B)[\cos A\cos B + \sin A\sin B - 2\cos A\cos B]$$

$$= \cos^2 B + \cos(A - B)(\sin A\sin B - \cos A\cos B)$$

$$= \cos^2 B - \cos(A - B)\cos(A + B)$$

$$= \cos^2 B - (\cos^2 A - \sin^2 B)$$

$$= \cos^2 B + \sin^2 B - \cos^2 A$$

$$= 1 - \cos^2 A$$

$$= \sin^2 A = \text{L.H.S.}$$

Exercise 4.4

1. Convert into a form of product :

(1) $\sin 7\theta + \sin 3\theta$ (2) $\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}$ (3) $\sin 3\theta - \sin 5\theta$

(4) $\sin\frac{7\theta}{2} - \sin\frac{3\theta}{2}$ (5) $\cos 11\theta + \cos 9\theta$ (6) $\cos\frac{5\theta}{2} + \cos\frac{11\theta}{2}$

(7) $\cos 5\theta - \cos 11\theta$ (8) $\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}$ (9) $\cos\theta - 1$

(10) $\sin\theta + 1$ (11) $\cos\theta + \sin\theta$ (12) $\sin\theta - \cos\theta$

Prove : **(2 to 7)**

2. (1) $\cos 55^\circ + \cos 65^\circ + \cos 175^\circ = 0$ (2) $\cos\frac{5\pi}{12} - \cos\frac{\pi}{12} = \frac{-1}{\sqrt{2}}$

(3) $\sin 65^\circ + \cos 65^\circ = \sqrt{2}\cos 20^\circ$ (4) $\dfrac{\sin\frac{5\pi}{12} - \cos\frac{5\pi}{12}}{\cos\frac{5\pi}{12} + \sin\frac{5\pi}{12}} = \frac{1}{\sqrt{3}}$

(5) $\dfrac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \cot A$

(6) $\cos 2\theta \cos\frac{\theta}{2} - \cos 3\theta \cos\frac{9\theta}{2} = \sin 5\theta \sin\frac{5\theta}{2}$

(7) $\sin\theta + \sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta + \frac{4\pi}{3}\right) = 0$

3. (1) $(\cos\alpha + \cos\beta)^2 + (\sin\alpha + \sin\beta)^2 = 4\cos^2\left(\frac{\alpha-\beta}{2}\right)$

(2) $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\left(\frac{\alpha-\beta}{2}\right)$

4. (1) $\sin A + \sin B + \sin C - \sin(A + B + C) = 4\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}$

(2) $\cos A + \cos B + \cos C + \cos(A + B + C) = 4\cos\frac{A+B}{2}\cos\frac{B+C}{2}\cos\frac{C+A}{2}$

5. (1) $\dfrac{\sin(A+B) - 2\sin A + \sin(A-B)}{\cos(A+B) - 2\cos A + \cos(A-B)} = \tan A$

(2) $\dfrac{\cos 3A + 2\cos 5A + \cos 7A}{\cos A + 2\cos 3A + \cos 5A} = \cos 2A - \sin 2A \tan 3A$

6. (1) $\dfrac{1}{\sin 10^\circ} - \dfrac{\sqrt{3}}{\cos 10^\circ} = 4$ (2) $\sqrt{2}\sin 10^\circ + \sqrt{3}\cos 35^\circ = \sin 55^\circ + 2\cos 65^\circ$

7. (1) $\sin\theta = n\sin(\theta + 2\alpha) \Leftrightarrow \tan(\theta + \alpha) = \frac{1+n}{1-n}\tan\alpha$

(2) $\sin(2A + 3B) = 5\sin B \Rightarrow 2\tan(A + 2B) = 3\tan(A + B)$

*

Miscellaneous Problems :

Example 16 : Prove that $0 < \alpha, \beta < \frac{\pi}{2} \Rightarrow \sin(\alpha + \beta) < \sin\alpha + \sin\beta$ and deduce from this that $\sin 49^\circ + \sin 41^\circ > 1$.

$$= -\frac{1 - sin\ 15^o}{cos\ 15^o}$$

$$= -\frac{1 - sin\ (45^o - 30^o)}{cos\ (45^o - 30^o)}$$

$$= -\frac{1 - \frac{\sqrt{3}-1}{2\sqrt{2}}}{\frac{\sqrt{3}+1}{2\sqrt{2}}}$$

$$= -\frac{2\sqrt{2} - \sqrt{3} + 1}{\sqrt{3} + 1}$$

$$= -\frac{(2\sqrt{2} - \sqrt{3} + 1)}{(\sqrt{3} + 1)} \frac{(\sqrt{3} - 1)}{(\sqrt{3} - 1)}$$

$$= -\frac{(2\sqrt{6} - 2\sqrt{2} - 3 + \sqrt{3} + \sqrt{3} - 1)}{2}$$

$$= -\sqrt{6} + \sqrt{2} + 2 - \sqrt{3}$$

$$= 2 + \sqrt{2} - \sqrt{3} - \sqrt{6}$$

Example 13 : If A + B + C = π, then prove that

$$sin\frac{A}{2} + sin\frac{B}{2} + sin\frac{C}{2} = 1 + 4sin\left(\frac{\pi - A}{4}\right) sin\left(\frac{\pi - B}{4}\right) sin\left(\frac{\pi - C}{4}\right).$$

Solution : R.H.S. $= 1 + 4sin\left(\frac{\pi - A}{4}\right) sin\left(\frac{\pi - B}{4}\right) sin\left(\frac{\pi - C}{4}\right)$

$$= 1 + 4sin\left(\frac{B + C}{4}\right) sin\left(\frac{A + C}{4}\right) sin\left(\frac{A + B}{4}\right) \quad (A + B + C = \pi)$$

$$= 1 + 2\left(2sin\left(\frac{B + C}{4}\right) sin\left(\frac{A + C}{4}\right)\right) sin\left(\frac{A + B}{4}\right)$$

$$= 1 + 2sin\left(\frac{A + B}{4}\right)\left[cos\left(\frac{B - A}{4}\right) - cos\left(\frac{A + B + 2C}{4}\right)\right]$$

$$= 1 + 2sin\left(\frac{A + B}{4}\right) cos\left(\frac{B - A}{4}\right) - 2sin\left(\frac{\pi - C}{4}\right) cos\left(\frac{\pi + C}{4}\right)$$

$$= 1 + \left(sin\frac{B}{2} + sin\frac{A}{2}\right) - \left(sin\frac{\pi}{2} - sin\frac{C}{2}\right)$$

$$= 1 + sin\frac{B}{2} + sin\frac{A}{2} - sin\frac{\pi}{2} + sin\frac{C}{2}$$

$$= sin\frac{A}{2} + sin\frac{B}{2} + sin\frac{C}{2} = \text{L.H.S.}$$

Example 14 : If α and β be the roots of the equation $acos\theta + bsin\theta = c$, prove that $tan\frac{\alpha}{2} + tan\frac{\beta}{2} = \frac{2b}{a + c}$. Hence, deduce that $tan\left(\frac{\alpha + \beta}{2}\right) = \frac{b}{a}$.

Solution : $acos\theta + bsin\theta = c$

$$\therefore \quad a\left(\frac{1 - tan^2\frac{\theta}{2}}{1 + tan^2\frac{\theta}{2}}\right) + b\left(\frac{2tan\frac{\theta}{2}}{1 + tan^2\frac{\theta}{2}}\right) = c$$

$$\therefore \quad a - atan^2\frac{\theta}{2} + 2btan\frac{\theta}{2} = c + ctan^2\frac{\theta}{2}$$

$$\therefore \quad (a + c)\ tan^2\frac{\theta}{2} - 2btan\frac{\theta}{2} + (c - a) = 0$$

Chapter 6

TRIGONOMETRIC EQUATIONS AND PROPERTIES OF A TRIANGLE

If equations are trains threading the landscape of numbers, then no train stops at pi.

– Richard Preston

Pure mathematics is in its way, the poetry of logical ideas.

– Albert Einstein

6.1 Introduction

In the previous semester and in chapters 4, 5 we have studied about trigonometric functions, their graphs and their properties like zeros, range, periodic nature, identities. Trigonometry is useful in land surveying. We know that by using trigonometry we can find the height of a hill without actually measuring it. In 1852, **Radhanath Sikdar**, an *Indian mathematician* and a surveyor from Bengal, was the first to identify Mount Everest as the world's highest peak, using trigonometric calculations. Trigonometry is useful in modern navigation such as satellite systems, astronomy, aviation, oceanography.

In this chapter we will learn how to solve trigonometric equations and properties of a triangle using trigonometry.

6.2 Trigonometric Equations

A trigonometric equation is an equation containing trigonometric functions, e.g. $sin^2x - 4cosx = 1$ is a trigonometric equation.

A trigonometric equation that holds true for all values of the variable in its domain is called a trigonometric identity, e.g. $cos2\theta = 2cos^2\theta - 1$ is a trigonometric identity.

There are other equations, which are true only for some proper subsets of domain of functions involved. We will learn some techniques for solving such trigonometric equations, as well as how to obtain the complete set of solutions of an equation based on a single solution of that equation. The equations $sinx = \frac{1}{2}$ has not only the solution $x = \frac{\pi}{6}$ but also $x = \frac{5\pi}{6}$, $x = 2\pi + \frac{\pi}{6}$, $x = 3\pi - \frac{\pi}{6}$ etc. are also solutions of $sinx = \frac{1}{2}$. Thus, we can say that $x = \frac{\pi}{6}$ is a solution of $sinx = \frac{1}{2}$ but it is not the complete solution of the equation. **A general solution to an equation is the set of all possible solutions of that equation.** Note that some trigonometric equations may not have any solution, e.g. $sinx = \pi$. Due to periodic nature of trigonometric functions, if a trigonometric equation has a solution it may have infinitely many solutions. The set of all such solution is known as the **general solution**.

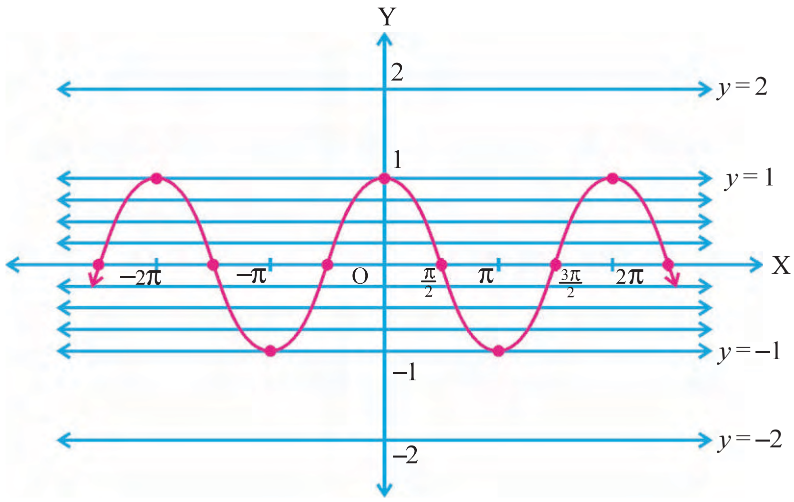

Figure 6.3

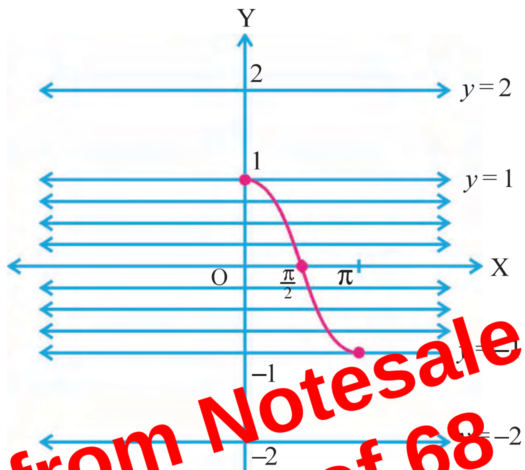

Figure 6.4

For the function $y = tanx$, if we draw any horizontal line in the plane it will intersect the graph of $y = tanx$ at infinitely many points (figure 6.5). This means that if we take any $a \in R$, then there are infinitely many real number x such that $tanx = a$. we need a unique value α such that $tan\alpha = a$. So we have to restrict domain suitably. We take $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ as restricted domain of $y = tanx$. (figure 6.6). We shall discuss this in more detail when we study the concept of inverse trigonometric functions in the third semester in 12th standard.

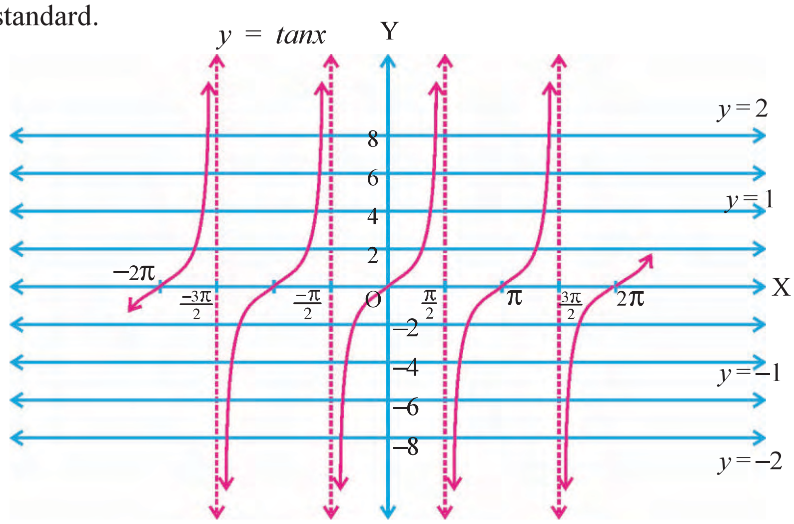

Figure 6.5

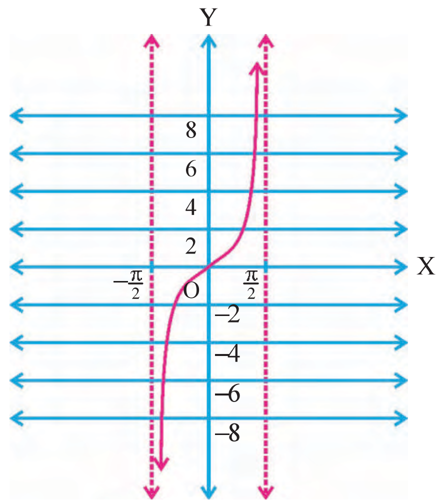

Figure 6.6

Thus, for any $a \in [-1, 1]$ there is a unique $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, such that, $a = \sin\alpha$.

Also, for any $a \in [-1, 1]$ there is a unique $\alpha \in [0, \pi]$, such that, $a = \cos\alpha$.

Finally, for any $a \in$ R there is a unique $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, such that, $a = \tan\alpha$.

We know the set of zeros of sine, cosine and tangent functions. That actually means that we already know the general solutions of the equations $\sin\theta = 0$, $\cos\theta = 0$, $\tan\theta = 0$.

$\sin\theta = 0 \Leftrightarrow \theta = k\pi, k \in Z$

$\cos\theta = 0 \Leftrightarrow \theta = (2k + 1)\frac{\pi}{2}, k \in Z$

$\tan\theta = 0 \Leftrightarrow \theta = k\pi, k \in Z$

We shall now solve the equations $\sin\theta = a, -1 \le a \le 1$, $\cos\theta = a, -1 \le a \le 1$ and $\tan\theta = a, a \in$ R.

6.3 General Solution of $\sin\theta = a$, where $-1 \le a \le 1$

Here $-1 \le a \le 1$. Therefore, there is a unique $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that, $a = \sin\alpha$.

Now, $\sin\theta = a = \sin\alpha$

$\therefore \quad \sin\theta - \sin\alpha = 0$

$\Leftrightarrow \quad 2\cos\frac{\theta+\alpha}{2} \sin\frac{\theta-\alpha}{2} = 0$

$\Leftrightarrow \quad \cos\frac{\theta+\alpha}{2} = 0$ or $\sin\frac{\theta-\alpha}{2} = 0$

$\Leftrightarrow \quad \frac{\theta+\alpha}{2} = (2n + 1)\frac{\pi}{2}$ or $\frac{\theta-\alpha}{2} = n\pi, n \in Z$ **(Why ?)**

$\Leftrightarrow \quad \theta = (2n + 1)\pi - \alpha$ or $\theta = 2n\pi + \alpha, n \in Z$

$\Leftrightarrow \quad \theta = (2n + 1)\pi + (-1)^{2n+1}\alpha$ or $\theta = 2n\pi + (-1)^{2n}\alpha, n \in Z$

Therefore, the general solution is given by $\theta = k\pi + (-1)^k\alpha, k \in Z$.

(We have replaced $2n + 1$, $2n$ by k because any integer is of the form either $2n + 1$ or $2n$)

Thus, **$\sin\theta = \sin\alpha \Leftrightarrow \theta = k\pi + (-1)^k\alpha, k \in Z$**

Hence, the solution set of $\sin\theta = a$, $-1 \le a \le 1$ is given by $\{k\pi + (-1)^k\alpha \mid k \in Z\}$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\theta = a = \sin\alpha$.

(We may take any $\alpha \in R$ such that $a = \sin\alpha$. The solution remains same. This convention of taking $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is only for the uniformity of the form of the solution set.)

General Solution of $\cos\theta = a$, where $-1 \le a \le 1$

Here $-1 \le a \le 1$. Therefore, there is a unique $\alpha \in [0, \pi]$ such that, $a = \cos\alpha$.

Now, $\cos\theta = a = \cos\alpha$

$$\therefore \quad \cos\theta - \cos\alpha = 0 \Leftrightarrow -2\sin\frac{\theta+\alpha}{2}\,\sin\frac{\theta-\alpha}{2} = 0$$

$$\Leftrightarrow \sin\frac{\theta+\alpha}{2} = 0 \text{ or } \sin\frac{\theta-\alpha}{2} = 0$$

$$\Leftrightarrow \frac{\theta+\alpha}{2} = k\pi \text{ or } \frac{\theta-\alpha}{2} = k\pi,\ k \in Z$$

$$\Leftrightarrow \theta = 2k\pi - \alpha \text{ or } \theta = 2k\pi + \alpha,\ k \in Z$$

Therefore the general solution is given by $\theta = 2k\pi \pm \alpha$, $k \in Z$.

Thus, **$\cos\theta = \cos\alpha \Leftrightarrow \theta = 2k\pi \pm \alpha$, $k \in Z$**

Hence, **the solution set of $\cos\theta = a$, $-1 \le a \le 1$ is given by $\{2k\pi \pm \alpha \mid k \in Z\}$ where $\alpha \in [0, \pi]$ and $\cos\theta = a = \cos\alpha$.**

General Solution of $\tan\theta = a$, where $a \in R$

Here $a \in R$. Therefore, there is a unique $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that, $a = \tan\alpha$.

Now, $\tan\theta = a = \tan\alpha$

$$\therefore \quad \tan\theta - \tan\alpha = 0 \Leftrightarrow \frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha} = 0$$

$$\Leftrightarrow \frac{\sin\theta\cos\alpha - \cos\theta\sin\alpha}{\cos\theta\cos\alpha} = 0$$

$$\Leftrightarrow \frac{\sin(\theta-\alpha)}{\cos\theta\cos\alpha} = 0$$

$$\Leftrightarrow \sin(\theta - \alpha) = 0$$

$$\Leftrightarrow \theta - \alpha = k\pi,\ k \in Z$$

$$\Leftrightarrow \theta = k\pi + \alpha,\ k \in Z$$

Thus, **$\tan\theta = \tan\alpha \Leftrightarrow \theta = k\pi + \alpha$, $k \in Z$**

Hence, **the solution set of $\tan\theta = a$, $a \in R$ is given by $\{k\pi + \alpha \mid k \in Z\}$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\tan\theta = a = \tan\alpha$.**

By the word 'solve' we shall mean to obtain the general solution set of the given equation.

Example 1 : Solve : (1) $2\sin 2\theta - 1 = 0$ (2) $\sin^2\theta - \sin\theta - 2 = 0$

Solution : (1) $2\sin 2\theta - 1 = 0$

$\therefore \quad \cos\theta = \cos\left(-\frac{\pi}{3}\right)$ and $\tan\theta = \tan\left(-\frac{\pi}{3}\right)$

$\therefore \quad \theta = 2k\pi - \frac{\pi}{3}, k \in Z$ **(P(θ) is in fourth quadrant.)**

Hence, required solution set is $\left\{2k\pi - \frac{\pi}{3} \;\middle|\; k \in Z\right\}$.

6.4 The General Solution of $a\cos x + b\sin x = c$, $a, b, c \in R$ and $a^2 + b^2 \neq 0$

For the given real numbers a and b, we can find $r > 0$ and $\alpha \in [0, 2\pi)$ such that $a = r\cos\alpha$ and $b = r\sin\alpha$. **(chapter 4)**

$\therefore \quad a^2 + b^2 = r^2\cos^2\alpha + r^2\sin^2\alpha = r^2$

$\therefore \quad r = \sqrt{a^2 + b^2}$ **($r > 0$)**

Now, $a\cos x + b\sin x = c$

$\therefore \quad r\cos\alpha\cos x + r\sin\alpha\sin x = c$

$\therefore \quad r\cos(x - \alpha) = c$

$\therefore \quad \cos(x - \alpha) = \frac{c}{r}$ **(i)**

The last equation will have a solution if and only if $\left|\frac{c}{r}\right| \leq 1$, that is if and only if $c^2 \leq r^2$, that is if and only if $c^2 \leq a^2 + b^2$.

If $\cos(x - \alpha) = \cos\beta$, where $\cos\beta = \frac{c}{r}$, $\beta \in [0, \pi]$, then the general solution of (i) is $x - \alpha = 2k\pi \pm \beta$, $k \in Z$ where $\alpha \in [0, 2\pi)$ such that $a = r\cos\alpha$ and $b = r\sin\alpha$.

Thus, if $c^2 \leq a^2 + b^2$, the general solution of $a\cos x + b\sin x = c$ is $x = 2k\pi + \alpha \pm \beta$, $k \in Z$, where $\alpha \in [0, 2\pi)$ such that $a = r\cos\alpha$ and $b = r\sin\alpha$ and $\cos\beta = \frac{c}{r}$, $\beta \in [0, \pi]$, $r = \sqrt{a^2 + b^2}$.

If $c^2 > a^2 + b^2$, the equation has no solution. In this case the solution set is $\emptyset$.

Example 6 : Solve : $\sqrt{3}\cos x + \sin x = \sqrt{2}$

Solution : Method 1 : Here $a = \sqrt{3}$, $b = 1$, $c = \sqrt{2}$.

$\therefore \quad r^2 = a^2 + b^2 = 3 + 1 = 4$

Hence, $r = 2$. Here $c^2 \leq a^2 + b^2$. So the given equation has a non-empty solution.

$a = r\cos\alpha$ and $b = r\sin\alpha$ gives $\cos\alpha = \frac{\sqrt{3}}{2}$ and $\sin\alpha = \frac{1}{2}$. Therefore $\alpha = \frac{\pi}{6}$

Now, $\cos\beta = \frac{c}{r} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\therefore \quad \beta = \frac{\pi}{4}$

Hence, required solution set is $\{2k\pi + \alpha \pm \beta \mid k \in Z\} = \left\{2k\pi + \frac{\pi}{6} \pm \frac{\pi}{4} \;\middle|\; k \in Z\right\}$.

8. $\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right) = 3$

9. $\sin x - 3\sin 2x + \sin 3x = \cos x - 3\cos 2x + \cos 3x$

10. $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0$

For ΔABC, prove **(11 to 14)** :

11. $a\cos A + b\cos B + c\cos C = 4R\sin A \sin B \sin C = \frac{abc}{2R^2}$

12. $a(\cos C - \cos B) = 2(b - c)\cos^2 \frac{A}{2}$

13. $a^3\cos(B - C) + b^3\cos(C - A) + c^3\cos(A - B) = 3abc$

14. $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} \Rightarrow \frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$

15. Prove : *cosine* rule using sine rule.

16. Prove : $(a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2} = c^2$

17. Prove : $abc(\cot A + \cot B + \cot C) = R(a^2 + b^2 + c^2)$

18. If length of the sides of a triangle are 4, 5 and 6, prove that the largest measure of an angle is twice that of the angle with smallest measure.

19. If length of the sides of a triangle are m, n, $\sqrt{m^2 + mn + n^2}$, prove that the largest measure of an angle of the triangle is $\frac{2\pi}{3}$.

20. If length of the two sides of a triangle are the roots of the equation $x^2 - 2\sqrt{3}x + 2 = 0$ and if the included angle between them has measure $\frac{\pi}{3}$, then show that the perimeter of the triangle is $2\sqrt{3} + \sqrt{6}$.

21. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) The set of values of x for which $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$ is ...

(a) $\emptyset$ (b) $\left\{\frac{\pi}{4}\right\}$

(c) $\left\{k\pi + \frac{\pi}{4} \mid k \in Z\right\}$ (d) $\left\{2k\pi + \frac{\pi}{4} \mid k \in Z\right\}$

(2) Number of ordered pairs (a, x) satisfying the equation $\sec^2 (a + 2)x + a^2 - 1 = 0$; $-\pi < x < \pi$ is ...

(a) 2 (b) 1 (c) 3 (d) infinite

(3) The general solution of the equation $\sin^{50}x - \cos^{50}x = 1$ is ...

(a) $2k\pi + \frac{\pi}{2}, k \in Z$ (b) $2k\pi + \frac{\pi}{3}, k \in Z$

(c) $k\pi + \frac{\pi}{3}, k \in Z$ (d) $k\pi + \frac{\pi}{2}, k \in Z$

(4) The number of solutions of the equation $3\sin^2x - 7\sin x + 2 = 0$, in the interval $[0, 5\pi]$ is ...

(a) 0 (b) 5 (c) 6 (d) 10