Chapter 4

PRINCIPLE OF MATHEMATICAL INDUCTION

Mathematics is the queen of science and number theory is the queen of mathematics.

– Gauss

Mathematics passes not only truth but also supreme beauty !

– Bertrand Russell

1.1 Introduction

We have studied one method of reasoning, deductive reasoning.

For example, consider the following statements :

(1) $1 + 2 + 3 + ... + 100 = 5050$

(2) $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

(3) Let $n = 100$ in (2). $1 + 2 + 3 + ... + 100 = \frac{(100)(101)}{2} = (50)(101) = 5050$

Here we want to prove that sum of all integers from 1 to 100 is 5050. We have a general result $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$. We take $n = 100$ in it and get the required result. Here, we apply a general principle to deduce a particular result.

Consider (1) If 3 divides product ab, then 3 divides a or 3 divides b. (2) If p is a prime and p divides ab then p divides a or p divides b. (3) Let $p = 3$ in (2) as 3 is a prime. Hence, if 3 divides product ab, then 3 divides a or 3 divides b.

Here also we apply a general principle to deduce a particular result.

(1) Amitabh Bachchan is a good actor.

(2) Actors are awarded national *Padma* honour in their category, if selected.

(3) Amitabh Bachchan was selected and got *Padma* honour.

Let $7x + 5y = k$ for $k \geq 24$, $x \in N \cup \{0\}$, $y \in N \cup \{0\}$. **(i)**

Now, $5 \cdot 3 - 7 \cdot 2 = 1$ **(ii)**

$\therefore \quad 7(x - 2) + 5(y + 3) = k + 1$ **(Adding (i) and (ii))**

Here $y + 3 \in N \cup \{0\}$ and $x - 2 \in N \cup \{0\}$ if $x \neq 0$ or 1.

Let $x = 0$. Then $5y = k \geq 24$. Thus $y \geq 5$, using (i).

$7 \cdot 3 - 5 \cdot 4 = 1$ and $5y = k$ gives on adding. **(iii)**

$7 \cdot 3 + 5(y - 4) = k + 1$

Here $x = 3 \geq 0$, $y - 4 \geq 0$ **($y \geq 5$)**

$\therefore \quad P(k + 1)$ is true, if $x = 0$

Let $x = 1$. Hence, $7 + 5y = k$, using (i).

Then $5y = k - 7 \geq 17$. Thus $y \geq 4$

$\therefore \quad 7 \cdot 3 - 5 \cdot 4 = 1$ and $7 + 5y = k$ gives on adding. **(iv)**

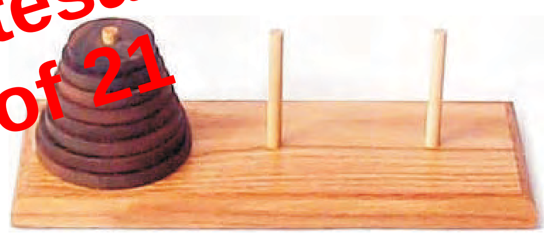
$7(4) + 5(y - 4) = k + 1$ with $y - 4 \geq 0$ and $x = 4$ **(Adding in (iv))**

$\therefore \quad P(k + 1)$ is true.

$\therefore \quad P(k)$ is true $\Rightarrow P(k + 1)$ is true.

$\therefore \quad P(n)$ is true for $\forall n \in N$ by P.M.I.

Example 22 : (Tower of Hanoi) We have three pegs and a collection of disks of different sizes. Initially they are on top on each other according to their size on the first peg, the largest being on the bottom and the smallest on the top. A move in this game consists of moving disks from one peg to another such that larger disk can never rest on a smaller one. Prove that the number of moves to transfer all disks from first peg to the last peg using the second peg as intermediate is $2^n - 1$, $n \in N$.



Solution : Let $P(n)$: The number of moves to transfer all disks from first peg to the last peg using the second peg as intermediate is $2^n - 1$, $n \in N$.

Let $n = 1$, obviously there is only one move.

$\therefore \quad P(1)$ is true. $2^1 - 1 = 1$. **($p(k)$)**

Suppose there are $2^k - 1$ moves to transfer k disks as required.

First we move top k disks to the second peg using the third peg as the intermediate one. This will take $2^k - 1$ moves. Now move the last disk to the third peg. This is one move. Now move k disks from second peg to the third peg in $2^k - 1$ moves.

$\therefore \quad$ The total number moves is $2^k - 1 + 1 + 2^k - 1 = 2 \cdot 2^k - 1 = 2^{k+1} - 1$

$\therefore \quad P(k + 1)$ is proved.

$\therefore \quad P(k)$ is true $\Rightarrow P(k + 1)$ is true.

$\therefore \quad P(n)$ is true, $\forall n \in N$ by P.M.I.

Example 23 : Prove $\frac{n^{11}}{11} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{62n}{165} \in N$, $n \in N$ (to be done after chapter 3)

Solution : Let P(n) : $\frac{n^{11}}{11} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{62n}{165} \in N$, $n \in N$

For $n = 1$, $\frac{n^{11}}{11} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{62n}{165} = \frac{15+33+55+62}{165} = \frac{165}{165} = 1$

∴ P(1) is true.

Let P(k) be true. Hence, $\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165} \in N$

Let $n = k + 1$.

Consider $\left(\frac{(k+1)^{11}}{11} + \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{62(k+1)}{165}\right) - \left(\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165}\right)$

$= \frac{1}{11}((k+1)^{11} - k^{11}) + \frac{1}{5}((k+1)^5 - k^5) + \frac{1}{3}((k+1)^3 - k^3) + \frac{62}{165}$

$= \frac{1}{11}\left(1 + \binom{11}{1}k + \binom{11}{2}k^2 + ... + \binom{11}{10}k^{10}\right) + \frac{1}{5}\left(1 + \binom{5}{1}k + \binom{5}{2}k^2 + ... + \binom{5}{4}k^4\right)$

$+ \frac{1}{3}\left(1 + \binom{3}{1}k + \binom{3}{2}k^2\right) + \frac{62}{165}$

$= \frac{1}{11}\binom{11}{1}k + \frac{1}{11}\binom{11}{2}k^2 + ... + \frac{1}{11}\binom{11}{10}k^{10} + \frac{1}{5}\binom{5}{1}k + \frac{1}{5}\binom{5}{2}k^2 + ... + \frac{1}{5}\binom{5}{4}k^4$

$+ \frac{1}{3}\binom{3}{1}k + \frac{1}{3}\binom{3}{2}k^2 + \frac{1}{11} + \frac{1}{5} + \frac{1}{3} + \frac{62}{165}$ **(i)**

Now, 11, 5, 3 being primes, 11 divides $\binom{11}{r}$ for $r = 1, 2, ..., 10$

5 divides $\binom{5}{r}$ for $r = 1, 2, 3, 4$

3 divides $\binom{3}{r}$ for $r = 1, 2$

and $\frac{1}{11} + \frac{1}{5} + \frac{1}{3} + \frac{62}{165} = 1$

∴ The R.H.S. in (1) represents a natural number.

Also $\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165} \in N$

∴ $\frac{(k+1)^{11}}{11} + \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{62(k+1)}{165}$

$= \frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165}$ + a natural number $\in N$

∴ P(k + 1) is true.

∴ P(k) is true ⟹ P(k + 1) is true.

∴ P(n) is true for $\forall n \in N$ by P.M.I.