(1) **Equality :** Two elements (a, b) and (c, d) of R $\times$ R are defined to be equal if $a = c$ and $b = d$. Thus $a = c,\ b = d \Rightarrow (a, b) = (c, d)$

For example, $(1, 0) = (sin^2x + cos^2x, \log 1)$ but $(1, 4) \neq (4, 1)$

(2) **Addition :** The sum of two elements (a, b) and (c, d) of R $\times$ R is defined as follows :

$(a, b) + (c, d) = (a + c, b + d)$

For example, $(5, 2) + (2, 3) = (5 + 2, 2 + 3) = (7, 5)$

(3) **Multiplication :** The product of two elements (a, b) and (c, d) of R $\times$ R is defined as follows :

$(a, b)(c, d) = (ac - bd, ad + bc)$

For example, $(5, 2)(2, 3) = (5 \times 2 - 2 \times 3, 5 \times 3 + 2 \times 2) = (4, 19)$

The set R $\times$ R with these rules is called the set of complex numbers and it is denoted by C. Generally, we denote a complex number by z.

2.3 Basic Algebraic Properties of Complex Numbers

We have discussed the properties of closure, commutativity, associativity and distributivity with respect to operations of addition and multiplication on R. We shall see that these properties hold good in C too.

The operation of addition satisfies the following properties :

(1) **The closure property :** The sum of two complex numbers is a complex number.

i.e. $z_1 + z_2 \in C \quad \forall z_1, z_2 \in C$

We also say that the addition is a binary operation on C.

(2) **The commutative property :** $z_1 + z_2 = z_2 + z_1 \quad \forall z_1, z_2 \in C$

(3) **The associative property :** $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \quad \forall z_1, z_2, z_3 \in C$

(4) **The existence of additive identity :** There exists a complex number O = (0, 0), called an additive identity or the zero complex number, such that

$z + O = z = O + z \quad \forall z \in C$

It can be proved that the additive identity O is unique.

In fact if $(a, b) + (x, y) = (a, b)$ for all $(a, b) \in$ C,

then $a + x = a, \qquad b + y = b,$

$\therefore\ x = 0, \qquad y = 0.$

Thus, $(x, y) = (0, 0)$

Also $(a, b) + (0, 0) = (a, b)$.

(5) **The existence of additive inverse :** To every complex number $z = (a, b)$, there corresponds a complex number $(-a, -b)$, denoted by $-z$, called the additive inverse (or negative) of z such that $z + (-a, -b) = (0, 0) = O$.

We observe that, $z + (-z) = (a, b) + (-a, -b)$

$= (a + (-a), b + (-b))$

$= (0, 0)$

$= O$ (O is the additive identity.)

Also, $(-z) + z = O$

We can prove that for $z \in$ C, its additive inverse $-z$ is unique.

Note : $(a, b) + (x, y) = (0, 0)$ requires $a + x = 0 = b + y$

$\therefore\ x = -a, y = -b$

2. $z + \overline{z} = a + ib + a - ib = 2a = 2Re(z)$ as $Re(z) = a$

$\therefore \quad \dfrac{z+\overline{z}}{2} = Re(z)$

3. $z - \overline{z} = a + ib - a + ib = 2ib = 2i\, Im(z)$ as $Im(z) = b$

$\therefore \quad \dfrac{z-\overline{z}}{2i} = Im(z)$

4. $z = \overline{z} \Leftrightarrow a + ib = a - ib \Leftrightarrow b = -b \Leftrightarrow 2b = 0 \Leftrightarrow b = 0.$

Thus, $z = \overline{z}$ if and only if z is real.

Modulus of a complex number :

Modulus of a complex number $z = a + ib$ is defined as $\sqrt{a^2+b^2}$ and is denoted by $|z|$.

Thus, $|z| = \sqrt{a^2+b^2}$

Note that $|z|$ is a real number and $|z| \geq 0, \quad \forall z \in$ C.

As an example, if $z = 3 + 4i$, then $|z| = \sqrt{9+16} = \sqrt{25} = 5$

Notice that if z is a real number (i.e. $z = a + 0i$) then, $|z| = \sqrt{a^2} = |a|$, where $|z|$ is the modulus of the complex number and $|a|$ is the absolute value of the real number (recall that for any real number a we have $\sqrt{a^2} = |a|$).

Properties of modulus :

1. $|z| = 0$ if and only if $z = 0$

2. $|z| \geq |Re(z)|, |z| \geq |Im(z)|$

3. $z\overline{z} = |z|^2$

4. $|z| = |\overline{z}|$

5. $|z| = |-z|$

6. $\dfrac{z_1}{z_2} = \dfrac{z_1\overline{z_2}}{|z_2|^2}$, where $z_2 \neq 0$

7. $|z_1 z_2| = |z_1||z_2|$

8. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ where $z_2 \neq 0$

9. $|z_1 + z_2| \leq |z_1| + |z_2|$ **(Triangular inequality)** (Why triangular ?)

10. $|z_1 - z_2| \geq ||z_1| - |z_2||$

Let us verify some of the above properties :

1. $|z| = 0 \Leftrightarrow \sqrt{a^2+b^2} = 0 \Leftrightarrow a^2 + b^2 = 0 \Leftrightarrow a = 0, b = 0 \Leftrightarrow z = 0$

2. $|z|^2 = a^2 + b^2 = (Re(z))^2 + (Im(z))^2 \geq (Re(z))^2$

$\therefore \quad |z| \geq |Re(z)|$ Similarly, $|z| \geq |Im(z)|$

3. $z\overline{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2$

4. $|z| = |a + ib| = \sqrt{a^2+b^2}$ and $|\overline{z}| = |a - bi| = \sqrt{a^2+(-b)^2} = \sqrt{a^2+b^2}$

so, $|z| = |\overline{z}|$

Geometrical representation of the sum of two complex numbers :

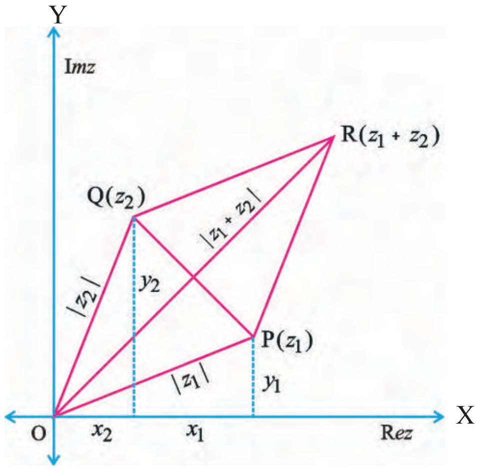

Figure 2.5

From the figure 2.5, in the argand plane P, Q and R represent z_1, z_2 and $z_1 + z_2$ respectively, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Mid-point of $\overline{OR}$ and $\overline{PQ}$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

$\therefore$ $\overline{OR}$ and $\overline{PQ}$ bisect each other.

Here, we have assumed that O, P and Q are non-collinear points.

The absolute values of z_1, z_2 and $z_1 + z_2$ are geometrically given by $|z_1| = OP$, $|z_2| = OQ = PR$ and $|z_1 + z_2| = OR$. We know that the sum of any two sides of a triangle is greater than the third side. Hence, in ΔORP, we have $OR < OP + PR$ implying $|z_1 + z_2| < |z_1| + |z_2|$. That is why this inequality for the absolute values of complex numbers is called the triangular inequality. (When does equality occur in $|z_1 + z_2| \leq |z_1| + |z_2|$?)

Polar representation of a complex number :

There is an alternate form to represent a complex number $z = x + iy$ which is known as polar representation. Let us understand how we can express any complex number into polar form. Let $z = x + iy$ be a non-zero complex number represented by the point P(x, y). (Figure 2.6). Draw $\overline{PM} \perp \overleftrightarrow{OX}$. Then OM = x and PM = y. Draw $\overline{OP}$. Let OP = r and $m\angle MOP = \theta$. Then $x = r\cos\theta$ and $y = r\sin\theta$.

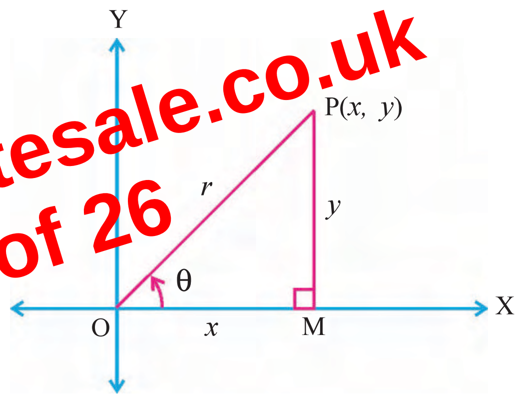

Figure 2.6

Therefore $z = x + iy = r(\cos\theta + i\sin\theta)$

Note : Here P lies in the first-quadrant.

$\therefore$ $x > 0$, $y > 0$. But if P(x, y) lies anywhere in the Argand plane except for origin, then also $x = r\cos\theta$, $y = r\sin\theta$ are true.

$\therefore$ $z = x + iy = r(\cos\theta + i\sin\theta)$

Here, $r^2 = x^2 + y^2$ **(r = OP > 0)**

$\therefore$ $r = \sqrt{x^2 + y^2}$ **($r > 0$)**

$\therefore$ $r = \sqrt{x^2 + y^2} = |z|$ and $\tan\theta = \frac{y}{x}$

The form $z = r(\cos\theta + i\sin\theta)$ is called the **polar form** of the complex number z. Also θ is known as **amplitude** or **argument of z**, written as **arg(z)**. Since *sine* and *cosine* functions are periodic, there are many values of θ satisfying $x = r\cos\theta$ and $y = r\sin\theta$. Each of these θ is an argument of z. The unique value of θ such that $-\pi < \theta \leq \pi$ for which $x = r\cos\theta$ and $y = r\sin\theta$ is known as the

(6) Let $z = -2i$. Here $z = 0 + iy$ and $y < 0$.

$\therefore$ Its polar form is $2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$.

Also, $|z| = 2$, $\arg z = \theta = -\frac{\pi}{2}$.

(7) Let $z = 1$. Here $z = x + i0$ and $x > 0$. So Its polar form is $1(\cos 0 + i\sin 0)$.

Also, $|z| = 1$, $\arg z = \theta = 0$.

(8) Let $z = 2i$. Here $z = 0 + iy$ and $y > 0$. So its polar form is $2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$.

Also, $|z| = 2$, $\arg z = \theta = \frac{\pi}{2}$.

Exercise 2.2

1. Find the absolute value and the principal argument of the following complex numbers :

(1) $\frac{1+7i}{(2-i)^2}$ (2) $\left(\frac{2+i}{3-i}\right)^2$ (3) $\sqrt{3} - i$ (4) $\frac{(1+i)(1+\sqrt{3}i)}{1-i}$ (5) $-3\sqrt{2} + 3\sqrt{2}\,i$

2. If $z = 3 + 2i$, then verify the following :

(1) $|z| = |\bar{z}|$ (2) $-|z| \le Re(z) \le |z|$ (3) $z^{-1} = \frac{\bar{z}}{|z|^2}$

3. If $z_1 = 3 + 2i$ and $z_2 = 2 - i$, then verify the following :

(1) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (2) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ (3) $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$ (4) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

4. If z is a non-zero complex number, show that $\overline{(z^{-1})} = (\bar{z})^{-1}$.

5. If $(a + ib)^2 = \frac{1+i}{1-i}$, show that $a^2 + b^2 = 1$.

6. If z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$, then is it necessary that $z_1 = z_2$? Justify your answer.

7. A complex number $z = a + ib$ is such that $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$. Show that $a^2 + b^2 - 2b = 1$.

8. Find the maximum value of $|1 + z + z^2 + z^3|$, if $z \in C$ and $|z| \le 3$.

9. (1) If $z = a + ib$ and $2|z - 1| = |z - 2|$, prove that $3(a^2 + b^2) = 4a$.

(2) If $z \in C$ such that $|2z - 3| = |3z - 2|$, prove that $|z| = 1$.

(3) If $z \in C$ such that $|2z - 1| = |z - 2|$, prove that $|z| = 1$.

10. Show that complex number $-3 + 2i$ is closer to the origin than $1 + 4i$.

11. Represent the points $-2 + 3i$, $-2 - i$ and $4 - i$ in the Argand diagram and prove that they are vertices of a right angled triangle.

12. Find the complex number z whose modulus is 4 and argument is $\frac{5\pi}{6}$.

13. If $(1 - 5i)z_1 - 2z_2 = 3 - 7i$, find z_1 and z_2, where z_1 and z_2 are conjugate complex numbers.

14. If $(a + ib)^2 = x + iy$ prove that $x^2 + y^2 = (a^2 + b^2)^2$.

15. If $\frac{(1+i)^2}{2-i} = x + iy$, then find the value of $x + y$.

*