Chapter 1: The Real Numbers $\mathbb{R}$

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Natural Numbers $\mathbb{N}$

$\mathbb{N} := \{ 1, 2, 3, 4, \dots \}$

Remark: $0 \notin \mathbb{N}$

For Addition:

$$n, m \in \mathbb{N} \rightsquigarrow m + n \in \mathbb{N}$$

Remark: "$\rightsquigarrow$" means "**leads to**"

For Multiplication:

$$n, m \in \mathbb{N} \rightsquigarrow mn \in \mathbb{N}$$

For Ordering:

$\forall m, n \in \mathbb{N}$ we must have that $m > n$ or $m < n$ or $m = n$

There are several properties of natural numbers (laws that must always be followed):

1. Commutative Law:
   i. $m + n = n + m$
   ii. $mn = nm$
2. Associative Law:
   i. $(m + n) + p = m + (n + p)$
   ii. $(mn)p = m(np)$
3. Distributive Law:
   i. $(m + n)p = mp + np$
4. Ordering:
   i. $m > n, \quad n > p, \quad$ then $\quad m > p$
   ii. $m > n \Rightarrow m + p > n + p$
   iii. $m > n, \quad p \in \mathbb{N} \Rightarrow mp > np$

Remark: Note that $p \in \mathbb{N}$ must be emphasized here.

There are some limitations to natural numbers.

Problem: Solving Equations

If $x + m = n$, then we know that $x = n - m$. We look into the problem, what if $m \geq n$? Then x is not a natural number. Therefore, we proceed to introduce *integers*.

Then now we need to prove that P_{k+1} is also true: $\qquad p_{k+1}^2 - 2q_{k+1}^2 = \pm 1$

$$\begin{aligned} LHS &= (p_k + 2q_k)^2 - 2(p_k + q_k)^2 \\ &= p_k^2 + 4p_kq_k + 4q_k^2 - 2p_k^2 - 4p_kq_k - 2q_k^2 \\ &= -p_k^2 + 2q_k^2 \\ &= -(p_k^2 - 2q_k^2) \\ &= \mp 1 \\ &= RHS \end{aligned}$$

Also, we have $q_{k+1} = p_k + q_k \geq k + 1$. And so, with our induction steps completed, we can safely conclude that $P_k \Rightarrow P_{k+1}$, and thus $\forall n \in \mathbb{N}$, we have $\left|\frac{p_n}{q_n} - \sqrt{2}\right| < \frac{1}{q_n^2} < \frac{1}{n^2}$, which is very small.

- **Attempt Homework 1**

Axioms for the Set of Real Numbers

Definition: Assume that X is a set where two binary operations are addition and multiplication. Where $X = \mathbb{R}$ and the two operations are the usual ones, then we say that X is a **field** if the following axioms are satisfied:

1) The axioms of arithmetic
2) The axioms of ordering
3) The completeness axiom

Axioms of Arithmetic:

These axioms hold for $\forall a, b, c \in X$

A1: $(a + b) + c = a + (b + c)$

A2: $a + b = b + a$

A3: $\exists 0 \in X$ s.t. $x + 0 = 0 + x = x \quad \forall x \in X$

A4: $\forall x \in X \quad \exists y \in X$ s.t. $x + y = 0 \quad$ (i.e. $y = -x$)

A5: $(a.b)c = a(b.c)$

A6: $a.b = b.a$

A7: $\exists 1 \in X$ s.t. $1.x = x.1 = x \quad \forall x \in X$

A8: $\forall x \in X, \quad x \neq 0, \; \exists y \in X$ s.t. $xy = 1 \quad$ (i.e $y = x^{-1}$)

A9: $x.(y + z) = x.y + x.z$

The axioms **A1** – **A4** mean that X is a commutative group w.r.t. *addition*. 0 is called the additive identity. The axioms **A5** – **A8** mean that $X \backslash \{0\}$ is a commutative group w.r.t *multiplication*. 1 is called the multiplicative identity.

Example: $x_n = \frac{1}{n}$. Prove that $x_n \to 0$.

Proof:

Given $\epsilon > 0$, we need to find $N \in \mathbb{R}$ s.t. $n > N \Rightarrow |x_n - l| < \epsilon$

$$\therefore \left|\frac{1}{n} - 0\right| < \epsilon$$

$$\Rightarrow n > \frac{1}{\epsilon}$$

As such, we let $N = \frac{1}{\epsilon}$, and so for any $n > N$ we have that $|x_n - 0| < \epsilon$. So from the definition of a limit we have that $x_n \to 0$. ■

> Remark: Any number N larger than $\frac{1}{\epsilon}$ would also be valid as the chosen N is the bare minimum for the requirement above. Therefore, we can also choose $N = 1 + \left[\frac{1}{\epsilon}\right] \in \mathbb{N}$. Therefore, N can be a natural number though it is not necessary.

Example: Let us prove that $x_n = \frac{1}{n} \nrightarrow 1$.

Proof:

x_n does not converge to l means that the definition of the limit where $x_n \to 1$ cannot be satisfied. Namely, negating the definition we get:

$$\exists \epsilon > 0 \text{ s.t. } \forall N \in \mathbb{R},\ \exists n > N \Rightarrow |x_n - l| \geq \epsilon$$

And so we begin with the same method, we let $l = 1$:

$$\left|\frac{1}{n} - 1\right| \geq \epsilon$$

$$\left|\frac{1 - n}{n}\right| \geq \epsilon$$

Now we need to consider two cases:

1. $\frac{3}{\epsilon} - 1 < 0$, then $n^2 > \frac{3}{\epsilon} - 1$ is always satisfied. As such, we can take N to be any number, and the definition of a limit will hold.
2. $\frac{3}{\epsilon} - 1 > 0$, then for $n^2 > \frac{3}{\epsilon} - 1 \Leftrightarrow n > \sqrt{\frac{3}{\epsilon} - 1}$. And so we just need to take $N = \sqrt{\frac{3}{\epsilon} - 1}$.

Overall, we can take $= \begin{cases} \mathbb{R} & , \frac{3}{\epsilon} - 1 < 0 \\ \sqrt{\frac{3}{\epsilon} - 1} & , \frac{3}{\epsilon} - 1 \geq 0 \end{cases}$, and this will satisfy the definition of a limit, and we have proof that $\frac{2n^2-1}{n^2+1} \to 2$. ■

Algebra of Limits

Now we have already shown how to prove a sequence converges to a claimed limit. But how do we identify that limit? (what to substitute for l)? Consider the previous example, $x_n = \frac{2n^2-1}{n^2+1}$. The limit is obtained as $n \to \infty$ if (x_n) converges. Therefore, to compute the l we can divide each term by the **dominant term** which is n^2: $x_n = \frac{2-\frac{1}{n^2}}{1+\frac{1}{n^2}}$ and we have $\frac{1}{n^2} \to 0$ as $n \to \infty$, and so $x_n \to 2$ by what we call the *algebra of limits*.

When computing limits, it is useful to identify **null sequences**, which are sequences which tend to 0 as $n \to \infty$. There are 3 forms of expressions a dominant term can take, namely: $\{n^p \ , \ a^n \ , \ n!\}$ in increasing order of dominance. Note that $p > 0$ and $|a| > 1$. To compute complicated limits we are usually required to identify to dominant term and divide all terms by it. This will procure some null sequences which we can use, along with the algebra of limits, to compute the overall limit of a sequence.

<u>Examples of null sequences:</u>

1. $\left(\frac{1}{n^p}\right)$, for $p > 0$
2. (c^n), for $|c| < 1$
3. $(n^p c^n)$, for $p > 0$ and $|c| < 1$
4. $\left(\frac{c^n}{n!}\right)$, for $c \in \mathbb{R}$
5. $\left(\frac{n^p}{n!}\right)$, for $p > 0$

Notice the null sequences are any term divided by the dominant term.

Theorem: Let $\langle x_n \rangle$, $\langle y_n \rangle$ and $\langle z_n \rangle$ be three sequences s.t. we have $x_n \le y_n \le z_n$ and if $x_n \to l$, $z_n \to l$ then $y_n \to l$. *This is known as the* ***sandwich theorem****.*

Proof:

Given $\epsilon > 0$, $\exists N_1 \in \mathbb{R}$ s.t. $n > N_1 \Rightarrow |x_n - l| < \epsilon$. This means that:

$$\Leftrightarrow -\epsilon < x_n - l < \epsilon$$
$$\Leftrightarrow l - \epsilon < x_n < l + \epsilon$$

Similarly, $\exists N_2 \in \mathbb{R}$ s.t. $n > N_2 \Rightarrow |z_n - l| < \epsilon$. Again this means that:

$$\Leftrightarrow l - \epsilon < z_n < l + \epsilon$$

Combining the two, we take $N = \max\{N_1, N_2\}$, and so for $n > N$ we have:

$$l - \epsilon < x_n \le y_n \le z_n < l + \epsilon$$
$$\Rightarrow l - \epsilon < y_n < l + \epsilon$$
$$\Rightarrow |y_n - l| < \epsilon$$

And so by definition of limits, we have $y_n \to l$. ■

Example: Consider $x_n = \frac{(-1)^n}{n^2} = \left\{-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, \dots\right\}$

Note that we cannot use the algebra of limits here without thinking. This is because $(-1)^n$ **does not converge**. This will be proven later on. However, using the sandwich theorem we can show that x_n converges.

$$\left|\frac{(-1)^n}{n^2}\right| = \frac{1}{n^2}$$

$$-\frac{1}{n^2} \le \frac{(-1)^n}{n^2} \le \frac{1}{n^2}$$

However since $\frac{1}{n^2} = \frac{1}{n} \times \frac{1}{n}$, we have that $\frac{1}{n^2} \to 0$ because $\frac{1}{n} \to 0$. Using the same reasoning, we have $-\frac{1}{n^2} \to 0$ also. By the sandwich theorem, $\frac{(-1)^n}{n^2} \to 0$ as well.

Series Tests

Here we will introduce tests which allow us to determine whether a series converges, without the need to compute partial sums.

Example: $\sum_{n=1}^{\infty}\frac{2^n}{3^n+17}$. We know that $\frac{2^n}{3^n+17} < \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$ and $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ converges (geometric series), and so $\sum_{n=1}^{\infty}\frac{2^n}{3^n+17}$ *should* converge.

Theorem: If $0 \le a_n \le b_n$ $\forall n \in \mathbb{N}$ and $\sum b_n$ converges, then $\sum a_n$ also converges with $\sum a_n \le \sum b_n$.

Corollary: If $0 \le a_n \le b_n$ $\forall n \in \mathbb{N}$ and $\sum a_n$ diverges, then $\sum b_n$ also diverges.

This is known as the ***comparison test****.*

Proof of theorem:

Denote the partial sums:

$$A_N = \sum_{n=1}^{N} a_n$$

$$B_N = \sum_{n=1}^{N} b_n$$

An important part of this proof is that since $a_n \ge 0$ $\forall n$, then the sequence of partial sums A_N is increasing ($S_{N+1} - S_N = a_{n+1} \ge 0$).

Given that $0 \le a_n \le b_n$, we know that $\langle A_N \rangle$ and $\langle B_N \rangle$ are increasing sequences. We know that $\langle B_N \rangle$ converges to $\sup B_N$ (due to its increasing property). Therefore:

$$B_\infty = \lim_{N\to\infty} B_N = \sup B_N = \sum_{n=1}^{\infty} b_n$$

But $A_N \le B_N$ since $a_n \le b_n$, $\forall n$. Thus, $A_N \le B_N \le \sup B_N = B_\infty$. So the sequence of partial sums $\langle A_N \rangle$ is increasing and bounded above, and therefore it must be convergent.

Thus, $\sum_{n=1}^{\infty} a_n$ converges. The limit of $\langle A_N \rangle$ is $\sup A_N \le \sup B_N$. ■

Proof (ii):

Assume $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = l > 1$, then we choose $\epsilon > 0$ s.t. $l - \epsilon > 1$. Then $\exists N \in \mathbb{N}$ s.t. $\forall n > N \Rightarrow \left|\frac{|a_{n+1}|}{|a_n|} - l\right| < \epsilon$, by definition of a limit.

$$\left|\frac{|a_{n+1}|}{|a_n|} - l\right| < \epsilon \Leftrightarrow l - \epsilon < \frac{|a_{n+1}|}{|a_n|} < l + \epsilon$$

$$\Rightarrow |a_{n+1}| > |a_n|(l - \epsilon)$$

Similar to the proof in (i) we consider $k \in \mathbb{N}$ s.t. $n \geq N + k$.

$$|a_{N+k}| \geq |a_{N+k-1}|(l-\epsilon) \geq |a_{N+k-2}|(l-\epsilon)^2 \ldots \geq |a_{N+1}|(l-\epsilon)^{k-1}$$

As $|a_{N+1}| > 0$ and $l - \epsilon > 1$, then as $k \to \infty$ we have $(l-\epsilon)^{k-1} \to \infty$, and as such $|a_{N+1}|(l-\epsilon)^{k-1} \to +\infty$. But we have:

$$|a_{N+1}|(l-\epsilon)^{k-1} \leq |a_{n+k}|$$

Therefore we have $|a_{N+k}| \to +\infty$ as $k \to \infty$.

We substitute in $n = N + k$ by definition, and so we obtain $\forall n > N$, $|a_n| \to +\infty$ as $n \to \infty$. As such, $a_n \nrightarrow 0$ so $\sum a_n$ diverges. ∎

Theorem: Suppose $\sqrt[n]{|a_n|} \to l \geq 0$ as $n \to \infty$, then:

i. $l < 1 \Rightarrow \sum a_n$ is absolutely convergent
ii. $l > 1 \Rightarrow \sum a_n$ is divergent and $|a_n| \to +\infty$
iii. $l = 1$ remains inconclusive

*This is known as the **Root Test**.*

The method of proof follows a similar concept to the ratio test.

However, what we are more interested in is the application of these tests. For instance, let us look at **power series**.

Example: $f(x) = 1 + x^2 \cos\left(\frac{1}{x}\right)$

Claim: $\lim_{x \to 0} f(x) = 1$

<u>Proof:</u>

Given $\epsilon > 0$, we need to find $\delta > 0$ s.t. $|x - 0| < \delta \Rightarrow |f(x) - 1| < \epsilon$.

$$|x| < \delta \Rightarrow \left|x^2 \cos\left(\frac{1}{x}\right) - 1\right| < \epsilon$$

Notice that $\forall x$, $\left|x^2 \cos\left(\frac{1}{x}\right)\right| \leq x^2$, so it is enough to achieve that $x^2 < \epsilon$. Therefore, take $\delta = \sqrt{\epsilon}$, so we have:

$$|x| < \delta \Rightarrow |x^2| < \delta^2 = \epsilon$$

$$|f(x) - 1| = \left|x^2 \cos\left(\frac{1}{x}\right) - l\right| \leq x^2 < \epsilon$$

And therefore, when $x \to 0$, $f(x) \to 1$. However, note that $f(x)$ **is not defined** at 0, though we need not consider this. ■

Notice as we tackle proofs as such, we need to pick δ skillfully such that we have the original bounding condition implying that $|f(x) - l| < \epsilon$. However, finding such δ is not always easy. Let us take a look at other examples, and their respective methods of proof.

Example: Consider the function $f(x) = \begin{cases} 2x & , \ x < 1 \\ 1 & , \ x = 1 \\ 4 - x & , \ x > 1 \end{cases}$

Claim: $\lim_{x \to 1^-} f(x) = 2$

<u>Proof:</u>

Given $\epsilon > 0$, we need to find $\delta > 0$ s.t. $x \in (1 - \delta, 1) \Rightarrow |f(x) - 2| < \epsilon$. We have:

$$\begin{aligned} x < 1 \ , |x - 1| < \delta \ &\Rightarrow |2x - 2| \\ &= 2|x - 1| < \epsilon \end{aligned}$$

This is as $x_n - c \to 0$, we have $\exp(x_n - c) \to 1$ because we have proved that $\exp(x)$ is continuous at $x = 0$ in step 1.

As such, $\exp(x_n) \to \exp(c)$ as $x_n \to c$, and so by definition of continuity, $\exp(x)$ is continuous on any $c \in \mathbb{R}$. ■

Theorem: Suppose g is continuous at $c \in \mathbb{R}$, and if f is continuous at $g(c)$, then $f \circ g$ is continuous at c. (Recall that $f \circ g = f\big(g(x)\big)$)

Proof:

Suppose $x_n \to c$, then $g(x_n) \to g(c)$ since g is continuous. Now since:

$$f \circ g(x_n) = f\big(g(x_n)\big) \to f\big(g(c)\big) = f \circ g(c)$$

$\Rightarrow f \circ g(x)$ is continuous at c. ■

Example: $\sqrt{\exp(x) + 17}$ is continuous.

Properties of Continuous Functions

Suppose $a, b \in \mathbb{R}$ and $f: [a, b] \to \mathbb{R}$ is continuous. If we want to study the **image** (or the range) of the function f, then we must look at the collection of all the values f will take.

$$f([a, b]) = \{f(x) | \, x \in [a, b]\}$$

Theorem 1: If $f: [a, b] \to \mathbb{R}$ is continuous, then f is bounded. This means that:

1. $\exists M = \sup f([a, b])$
2. $\exists m = \inf f([a, b])$

Theorem 2: Every continuous function $f: [a, b] \to \mathbb{R}$ attains a maximum and minimum:

$$\exists x_{\max}, x_{\min} \in [a, b] \; s.t. \, M = f(x_{\max}) \text{ and } m = f(x_{\min})$$

And $f(x_{\min}) = m \leq f(x) \leq M = f(x_{\max}) \; \forall x \in [a, b]$.

Notice that we are dealing with a **closed interval** $[a, b]$ here. $f: [a, b] \to \mathbb{R}$ is continuous if

i. At any point $c \in (a, b)$ we have $\lim_{x \to c} f(x) = f(c)$

ii. $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$

<u>Proof of Theorem 1:</u>

Assume f is not bounded, say f is unbounded above. Then this means that $\nexists M \in \mathbb{R}$ s.t. $f(x) \leq M \ \forall x \in [a, b]$ by using the definition of an upper bound. In particular, $\forall n \in \mathbb{N}$, $\exists x = x_n \in [a, b]$ s.t. $f(x_n) > M$.

Therefore, based on our assumption that f is unbounded, as $n \to +\infty$, then $f(x_n) \to +\infty$.

Since $x \in [a, b]$, then we know $\langle x_n \rangle$ is bounded, we can use the **bolzano-weiestrass** theorem and find a convergent subsequence. So let this subsequence be $\langle x_{j_n} \rangle$:

$$x_{j_n} \to x_\infty \ , \ \ x_\infty = \lim(x_{j_n})$$

Claim: $x_\infty \in [a, b]$

<u>Proof of claim:</u>

Suppose that this is not true. Then either $x_\infty < a$ or $x_\infty > b$. That means x_∞ lies outside the interval $[a, b]$.

Now suppose $x_\infty < a$, then because $x_{j_n} \to x_\infty$, $\forall \epsilon > 0$, for sufficiently large n, we have x_{j_n} lies in the ϵ-neighbourhood of x_∞ by definition of a limit of a sequence.

$$x_{j_n} \in (x_\infty - \epsilon, x_\infty + \epsilon)$$

If we choose a small $\epsilon > 0$ s.t. $x_\infty + \epsilon < a$, this contradicts the statement that $x_{j_n} \in [a, b]$. Therefore, $x_\infty \in [a, b]$. ■

And so, we know that $x_\infty \in [a, b]$. Since we are given that f is **continuous** within $[a, b]$, then it is continuous at x_∞, and by definition of continuity we have $f(x_{j_n}) \to f(x_\infty)$. But $f(x_\infty)$ is supposed to be a real number, and therefore this contradicts that $f(x_n) \to +\infty$.

Therefore, f is bounded. ■

Proposition: $\exp(x)$ *is a bijection from* $\mathbb{R}$ *to* $(0, +\infty)$

Proof:

Suppose $x' > x$, then $x' = x + h, h > 0$.

Therefore, $\frac{\exp(x')}{\exp(x)} = \frac{\exp(x)\exp(h)}{\exp(x)} = \exp(h) > 1$ because $h > 0$.

So this implies that $\exp(x') > \exp(x)$. Thus, exp is strictly increasing and therefore, exp is **injective**, meaning that $\exp(x) = y$ where y *cannot have more than one solution*. We now need to show that $\exp(x)$ is surjective.

Claim: exp is surjective on the interval $(0, +\infty)$. This means that $\forall y > 0$*, there is at least one solution for* x *for which* $\exp(x)$ *equals to a particular value* y*.* (We use the intermediate value theorem)

Proof of Claim:

Since $\exp(x) \geq 1 + x$, we have for $x > 0$:

$$\exp(x) > 1$$
$$\exp(0) = 1$$

And for $x < 0$ we have:

$$\exp(x) = \frac{1}{\exp(-x)} \in (0,1)$$

Therefore, because we know $\lim_{x\to+\infty} \exp(x) = +\infty$, then $\exists h \in \mathbb{R}$ s.t. $\forall x > h$, we have $\exp(h) \geq \bar{y} \Rightarrow \exp(x) > \bar{y}$. In particular, $\exp(h + 1) > \bar{y}$.

We also know that since $\exp(x) = \frac{1}{\exp(-x)} \to 0$ as $x \to -\infty$. Therefore, $\exists h \in \mathbb{R}$ s.t. $\forall x < h$, we have $\exp(h) \leq \bar{y} \Rightarrow \exp(x) < \bar{y}$. In particular, $\exp(h - 1) < \bar{y}$.

By the intermediate value theorem,

$$\exists \bar{x} \in [h - 1, h + 1] \text{ s.t. } \exp(\bar{x}) = \bar{y} \textbf{ (surjection)}$$

Therefore, $\exp: \mathbb{R} \to (0, +\infty)$ is a bijection. ∎

4) $f(x) = x^{17}$,

$$f'(c) = \lim_{h\to 0} \frac{(c+h)^{17} - c^{17}}{h}$$
$$= \lim_{h\to 0} \frac{\left(\sum_{r=0}^{17} c^r h^{17r}\right) - c^{17}}{h}$$
$$= \lim_{h\to 0} \frac{c^{17} + 17c^{16}h + h^2\binom{17}{2}c^{15} \ldots - c^{17}}{h}$$
$$= \lim_{h\to 0} \frac{17c^{16}h}{h} = 17c^{16}$$

Therefore, $f'(x) = 17x^{16}$

5) $f(x) = \frac{1}{x}$, $x \neq 0$

$$f'(c) = \lim_{h\to 0} \frac{\frac{1}{c+h} - \frac{1}{c}}{h}$$
$$= \lim_{h\to 0} \frac{\left(\frac{c-c-h}{c(c+h)}\right)}{h}$$
$$= \lim_{h\to 0} \frac{-1}{c(c+h)}$$
$$= -\frac{1}{c^2}$$

Therefore, $f'(x) = -\frac{1}{x^2}$

In general, $(x^n) = nx^{n-1}$ (to be proved later).

6) $f(x) = |x| = \begin{cases} x \ , x>0 \\ -x \ , x<0 \end{cases}$

Let $c = 0$. Then $\frac{f(c+h)-f(c)}{h} = \frac{|h|}{h} = \begin{cases} 1 \ , h>0 \\ -1 \ , h<0 \end{cases}$

Then there is no limit as $h \to 0$ so $|x|$ is not differentiable at $x = 0$.

We know that $e^x \geq 1 + x$ and that $e^x = \frac{1}{e^{-x}} \leq \frac{1}{1-x}$, $x < 1$, so:

$$1 + x \leq e^x \leq \frac{1}{1-x} \quad , \quad -1 < x < 1$$

Now how do we find the derivative of e^x at $x = 0$?

Theorem: Suppose $p(x)$ and $q(x)$ are differentiable at $x = c$, and

$$p(c) = q(c) = L$$
$$p'(c) = q'(c) = m$$

Assume that:

$$p(x) \leq f(x) \leq q(x)$$

Holds in a neighbourhood of c. Then f is differentiable at c and $f'(c) = m$.

This is known as the ***Sandwich Theorem for derivatives***.

<u>Proof:</u>

We have $L = p(c) \leq f(c) \leq q(c) = L$, so $f(c) = L$. Therefore,

$$p(c+h) - L \leq f(c+h) - f(c) \leq q(c+h) - L$$
$$p(c+h) - p(c) \leq f(c+h) - f(c) \leq q(c+h) - q(c)$$

Suppose $h > 0$, then:

$$\frac{p(c+h) - p(c)}{h} \leq \frac{f(c+h) - f(c)}{h} \leq \frac{q(c+h) - q(c)}{h}$$

As $\lim_{h \to 0^+} \frac{p(c+h)-p(c)}{h} = p'(c) = \lim_{h \to 0^+} \frac{q(c+h)+q(c)}{h} = q'(c) = m$. So $\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} = m$ via the sandwich theorem for limits.

Theorems for Differentiable Functions

Theorem: Suppose $a < b$, and $f: [a, b] \to \mathbb{R}$, f is continuous on $[a, b]$ and differentiable on (a, b). Assume also that $f(b) = f(a)$, then:

$$\exists c \in (a, b) \text{ s.t. } f'(c) = 0$$

*This is known as **Rolle's Theorem**.*

Proof:

f is continuous on $[a, b]$ $\Rightarrow$ f attains its maximal and minimal values

$\Rightarrow \exists$ a global maximum $x_{\max}$ and a global minimum $x_{\min}$.

Case 1:

$f(x_{\max}) = f(x_{\min})$

Then $f(x)$ is constant and $f'(x) = 0 \ \forall x$. Take arbitrary $c \in (a, b)$.

Case 2:

$f(x_{\min}) < f(x_{\max})$

Then at least one of the numbers $f(x_{\max})$ or $f(x_{\min})$ is different from $f(a) = f(b)$.

Thus, $x_{\max}$ or $x_{\min}$ are inside (a, b), so this is a **local extremum**, so $f'(x_{\max}) = 0$ or $f'(x_{\min}) = 0$. ■