Prove uniqueness (or non-uniqueness) of the Heat equation. [5]

[1] Let T_1, T_2 be solutions of the IBVP $T_t = \kappa T_{xx}$ for 0 < x < L, t > 0 subject to initial condition $T_i(x,0) = f(x)$ on $0 \le x \le L$, and boundary conditions $T_i(0,t) = T_i(L,t) = 0$ for t > 0. Consider the difference $D := T_1 - T_2$. Then D satisfies

$$D_t = \kappa D_{xx}, \quad 0 < x < L, \ t > 0$$
$$D(x, 0) = 0, \quad 0 \le x \le L,$$

and boundary conditions

$$D(0,t) = D(L,t) = 0, \quad t > 0.$$

[2] Let

$$I(t) = \frac{1}{2} \int_0^L [D(x,t)]^2 \,\mathrm{d}x.$$

Then $I(t) \ge 0$ and I(0) = 0. By Leibniz's rule, the derivative of I,

$$I'(t) = \int_0^L DD_t \, \mathrm{d}x = \kappa \int_0^L DD_{xx} \, \mathrm{d}x = \kappa \int_0^L \left[(DD_x)_x - D_x^2 \right] \, \mathrm{d}x,$$

the last equality coming from using the product rule: $(DD_x)_x = D_x D_x + DD_{xx}$.

[1] Now carrying out the integration and using both BCs, we see

- so I cannot increase.
- [1] Hence

for every $t \ge 0$ and thus W = 0, i.e. $T_1 = T_2$ (uniqueness).

Note:

• UNIQUENESS PROOFS always start by considering a difference D between two solutions T_1, T_2 .

1

Derive the one-dimensional Wave equation... [6]

Stating your assumptions, derive the wave equation for the small transverse displacements y(x,t) of an elastic string of constant density ρ , under constant tension T, where $c = \sqrt{T/\rho}$.

We start by assuming that $|y_x| \ll 1$ and ignoring gravity and air resistance.

[1] A point initially at $x\mathbf{i}$ is displaced to $\mathbf{r}(x,t) = x\mathbf{i} + y(x,t)\mathbf{j}$. The vector $\mathbf{t} := \mathbf{r}_x = \mathbf{i} + y_x\mathbf{j}$ is a tangent vector to the string. Since

$$|\mathbf{t}| = \sqrt{1+y_x^2} = 1 + \frac{1}{2}y_x^2 - \frac{1}{8}y_x^4 + \cdots,$$

it is approximately a unit tangent from our assumption.

[1] Hence $T\mathbf{t}$ is the force exerted by + on - (and the other way round for $-T\mathbf{t}$). The velocity and acceleration vectors are

$$\mathbf{v} = \mathbf{r}_t = y_t \mathbf{j}$$
 and $\mathbf{a} = \mathbf{r}_{tt} = y_{tt} \mathbf{j}$.

[1] Consider an interval [a, a + h]. We have

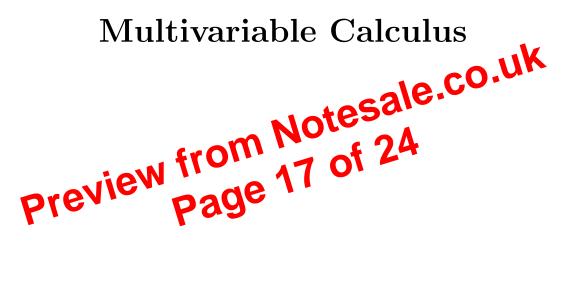
net force =
$$T\mathbf{t}(a+h,t) - T\mathbf{t}(a,t)$$

momentum = $\int_{0}^{a+h} \rho \phi \mathbf{s} \mathbf{s} \mathbf{d} \mathbf{s}$.
[2] By using Newton's Second Lier, then Leibniz' rule of letting $h \to 0$ (note $\mathbf{v}_t = \mathbf{a}$),
prevent force = consolid change of momentum
 $\Rightarrow T\mathbf{t}(a+h,t) - T\mathbf{t}(a,t) = \frac{d}{dt} \int_{a}^{a+h} \rho \mathbf{v}(x,t) dx$
 $\Rightarrow T\left(\frac{\mathbf{t}(a+h,t) - \mathbf{t}(a,t)}{h}\right) = \frac{1}{h} \int_{a}^{a+h} \rho \mathbf{a}(x,t) dx$
 $\Rightarrow T\mathbf{t}_x(a,t) = \rho \mathbf{a}(a,t).$

[1] Substitute for \mathbf{t}_x and \mathbf{a} in terms of y(x,t) to get

$$Ty_{xx}\mathbf{j} = \rho y_{tt}\mathbf{j} \Rightarrow y_{tt} = c^2 y_{xx}.$$

Part II



Derive Gauss' Flux Theorem using Poisson's equation. [4]

[1] For a scalar gravitational potential $\phi(\mathbf{r})$, Poisson's equation reads

$$\nabla^2 \phi(\mathbf{r}) = -4\pi G \rho(\mathbf{r}).$$

[1] We want to derive Gauss' Flux Theorem, which states that for a smooth and bounded region R, which contains matter of total mass M, we have

$$\iint_{\partial R} \mathbf{f} \cdot \mathrm{d}\mathbf{S} = -4\pi G M.$$

[1] Apply the Divergence Theorem to the LHS and use the fact that $\mathbf{f} = \nabla \phi$, since ϕ is a potential:

$$\iint_{\partial R} \mathbf{f} \cdot \mathrm{d}\mathbf{S} = \iiint_R \nabla \cdot \mathbf{f} \,\mathrm{d}V = \iiint_R \nabla^2 \phi \,\mathrm{d}V$$

[1] and by Poisson's equation and the definition of total mass we get

$$= -4\pi G \iiint_R \rho \, \mathrm{d}V$$

=-4\pi GM. Preview from Notesale.co.uk page 24 of 24