Prove uniqueness (or non-uniqueness) of the Heat equation. [5]

[1] Let T_1, T_2 be solutions of the IBVP $T_t = \kappa T_{xx}$ for $0 < x < L$, $t > 0$ subject to initial condition $T_i(x, 0) = f(x)$ on $0 \le x \le L$, and boundary conditions $T_i(0, t) = T_i(L, t) = 0$ for $t > 0$. Consider the difference $D := T_1 - T_2$. Then D satisfies

$$D_t = \kappa D_{xx}, \quad 0 < x < L,\, t > 0$$

$$D(x, 0) = 0, \quad 0 \le x \le L,$$

and boundary conditions

$$D(0, t) = D(L, t) = 0, \quad t > 0.$$

[2] Let

$$I(t) = \frac{1}{2}\int_0^L [D(x,t)]^2 \,\mathrm{d}x.$$

Then $I(t) \ge 0$ and $I(0) = 0$. By Leibniz's rule, the derivative of I,

$$I'(t) = \int_0^L DD_t \,\mathrm{d}x = \kappa \int_0^L DD_{xx} \,\mathrm{d}x = \kappa \int_0^L \left[(DD_x)_x - D_x^2\right] \mathrm{d}x,$$

the last equality coming from using the product rule: $(DD_x)_x = D_x D_x + DD_{xx}$.

[1] Now carrying out the integration and using both BCs, we see

$$I'(t) = -\kappa \int_0^L D_x^2 \,\mathrm{d}x \le 0,$$

so I cannot increase.

[1] Hence

$$0 \le I(t) \le I(0) = 0 \;\Rightarrow\; I(t) = 0$$

$$\Rightarrow \int_0^L [D(x,t)]^2 \,\mathrm{d}x = 0$$

for every $t \ge 0$ and thus $W = 0$, i.e. $T_1 = T_2$ (uniqueness).

Note:

- **UNIQUENESS PROOFS** always start by considering a difference D between two solutions T_1, T_2.

Derive the one-dimensional Wave equation... [6]

Stating your assumptions, derive the wave equation for the small transverse displacements $y(x,t)$ of an elastic string of constant density ρ, under constant tension T, where $c = \sqrt{T/\rho}$.

We start by assuming that $|y_x| \ll 1$ and ignoring gravity and air resistance.

[1] A point initially at $x\mathbf{i}$ is displaced to $\mathbf{r}(x,t) = x\mathbf{i} + y(x,t)\mathbf{j}$. The vector $\mathbf{t} := \mathbf{r}_x = \mathbf{i} + y_x\mathbf{j}$ is a tangent vector to the string. Since

$$|\mathbf{t}| = \sqrt{1+y_x^2} = 1 + \frac{1}{2}y_x^2 - \frac{1}{8}y_x^4 + \cdots,$$

it is approximately a unit tangent from our assumption.

[1] Hence $T\mathbf{t}$ is the force exerted by $+$ on $-$ (and the other way round for $-T\mathbf{t}$). The velocity and acceleration vectors are

$$\mathbf{v} = \mathbf{r}_t = y_t\mathbf{j} \quad \text{and} \quad \mathbf{a} = \mathbf{r}_{tt} = y_{tt}\mathbf{j}.$$

[1] Consider an interval $[a, a+h]$. We have

$$\begin{aligned} \text{net force} &= T\mathbf{t}(a+h,t) - T\mathbf{t}(a,t) \\ \text{momentum} &= \int_a^{a+h} \rho\mathbf{v}(x,t)\,\mathrm{d}x. \end{aligned}$$

[2] By using Newton's Second Law, then Leibniz' rule and letting $h \to 0$ (note $\mathbf{v}_t = \mathbf{a}$),

$$\begin{aligned} &\text{net force} = \text{rate of change of momentum} \\ &\Rightarrow T\mathbf{t}(a+h,t) - T\mathbf{t}(a,t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_a^{a+h} \rho\mathbf{v}(x,t)\,\mathrm{d}x \\ &\Rightarrow T\left(\frac{\mathbf{t}(a+h,t) - \mathbf{t}(a,t)}{h}\right) = \frac{1}{h}\int_a^{a+h} \rho\mathbf{a}(x,t)\,\mathrm{d}x \\ &\Rightarrow T\mathbf{t}_x(a,t) = \rho\mathbf{a}(a,t). \end{aligned}$$

[1] Substitute for $\mathbf{t}_x$ and $\mathbf{a}$ in terms of $y(x,t)$ to get

$$Ty_{xx}\mathbf{j} = \rho y_{tt}\mathbf{j} \Rightarrow y_{tt} = c^2 y_{xx}.$$

Part II

Multivariable Calculus

Derive Gauss' Flux Theorem using Poisson's equation. [4]

[1] For a scalar gravitational potential $\phi(\mathbf{r})$, Poisson's equation reads

$$\nabla^2\phi(\mathbf{r}) = -4\pi G\rho(\mathbf{r}).$$

[1] We want to derive Gauss' Flux Theorem, which states that for a smooth and bounded region R, which contains matter of total mass M, we have

$$\iint_{\partial R} \mathbf{f} \cdot \mathrm{d}\mathbf{S} = -4\pi GM.$$

[1] Apply the Divergence Theorem to the LHS and use the fact that $\mathbf{f} = \nabla\phi$, since ϕ is a potential:

$$\iint_{\partial R} \mathbf{f} \cdot \mathrm{d}\mathbf{S} = \iiint_R \nabla \cdot \mathbf{f}\, \mathrm{d}V = \iiint_R \nabla^2\, \phi\, \mathrm{d}V$$

[1] and by Poisson's equation and the definition of total mass we get

$$= -4\pi G \iiint_R \rho\, \mathrm{d}V$$

$$= -4\pi GM.$$