Now choose h, k such that $a_1 h + b_1 k + c_1 = 0$

& $a_2 h + b_2 k + c_2 = 0$.

$$\Rightarrow \frac{dY}{dX} = \frac{a_1 X + b_1 Y}{a_2 X + b_2 Y} \Big\} \Rightarrow \text{homogeneous}$$

put $Y = VX$.

$$\Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}.$$ & VOBS.

→ Linear DE.

it is of the form $\frac{dy}{dx} + P(x)y = Q(x)$.

$$IF \Rightarrow e^{\int P(x)\,dx}$$

General soln $\Rightarrow y(IF) = \int Q(x) \cdot IF + c$

→ Bernoulli's DE.

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

multiply by y^{-n} on both sides.

$$\Rightarrow y^{-n}\frac{dy}{dx} + P(x)y^{1+n} = Q(x).$$

put $y^{1+n} = t$.

$$(1+n)y^{-n}\frac{dy}{dx} = \frac{dt}{dx}.$$

$$\Rightarrow y^{-n}\frac{dy}{dx} = \frac{1}{(1+n)}\frac{dt}{dx}.$$

$$\Rightarrow \cancel{\frac{1}{(1+n)}}\frac{dt}{dx} + (1+n)P(x)t = Q(x)(1+n).$$

↳ L.D.E.

Differentiation

→ According to first principle of derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Some common derivatives

→ $\frac{d}{dx}(k) = 0$ → $\frac{d}{dx}(kx) = k$

→ $\frac{d}{dx}(x^n) = nx^{n-1}$ → $\frac{d}{dx}(e^x) = e^x$

→ $\frac{d}{dx}(a^x) = a^x \log a$ → $\frac{d}{dx}(\log x) = \frac{1}{x}$

→ $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$ → $\frac{d}{dx}(\sin x) = \cos x$

→ $\frac{d}{dx}(\cos x) = -\sin x$ → $\frac{d}{dx}(\tan x) = \cancel{\cot x}\ \sec^2 x$

→ $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$ → $\frac{d}{dx}(\sec x) = \sec x \tan x$

→ $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$ → $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

→ $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$ → $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

→ $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$ → $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

→ $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$ → $\frac{d}{dx}(\sinh x) = \cosh x$

→ $\frac{d}{dx}(\cosh x) = \sinh x$ → $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$

→ $\frac{d}{dx}(\coth x) = -\text{cosech}^2 x$ → $\frac{d}{dx}(\text{sech}\, x) = \text{sech}\, x \tanh x$

→ $\frac{d}{dx}(\text{cosech}\, x) = -\text{cosech}\, x \coth x$.

→ $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ put $x = \tan\theta$.

→ $\sqrt{a^2-x^2} \Rightarrow x = a\sin\theta$ (or) $a\cos\theta$.

→ $\sqrt{x^2+a^2} \Rightarrow x = a\tan\theta$ (or) $a\cot\theta$.

→ $\sqrt{x^2-a^2} \Rightarrow x = a\sec\theta$ (or) $a\,\mathrm{cosec}\,\theta$.

→ $\sqrt{\frac{a-x}{a+x}} \Rightarrow x = a\cos 2\theta$.

→ $\sqrt{\frac{a^2-x^2}{a^2+x^2}} \Rightarrow x^2 = a^2\cos 2\theta$.

→ $\sqrt{ax-x^2} \Rightarrow x = a\sin^2\theta$.

→ $\sqrt{\frac{x}{a+x}} \Rightarrow x = a\tan^2\theta$.

→ $\sqrt{\frac{x}{a-x}} \Rightarrow x = a\sin^2\theta$.

→ $\sqrt{(x-a)(x-b)} \Rightarrow x = a\sec^2\theta - b\tan^2\theta$.

→ $\sqrt{(x-a)(b-x)} \Rightarrow x = a\cos^2\theta + b\sin^2\theta$.

Differentiation of ∞ series

→ $y = f(x)^{f(x)^{f(x)\cdots\infty}}$ $\Rightarrow y = f(x)^y$. Apply log.

$\log y = y\log f(x)$.

Differentiate O.B.S.

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log f(x)\cdot\frac{dy}{dx} + \frac{y}{f(x)}\cdot f'(x).$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y} - \log f(x)\right) = \frac{y}{f(x)}\cdot f'(x).$$

→ $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \ldots \infty}}}$

$\Rightarrow y = \sqrt{f(x) + y}$

$\Rightarrow y^2 = f(x) + y$ D.O.B.S.

→ $y = f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x) + \cfrac{1}{f(x) + \ldots}}}$

$$\frac{dy}{dx} = \frac{y f'(x)}{2y - f(x)}.$$

Functions

(Domain, Range & Graphs).

Function	Domain	Range
x^n	R	R if n is odd $R^+ \cup \{0\}$ if n is even.
$\frac{1}{x^n}$	$R - \{0\}$	$R - \{0\}$ if n is odd. R^+ if n is even
$x^{1/n}$	R if n is odd $R^+ \cup \{0\}$ if n is even	R if n is odd. $R^+ \cup \{0\}$ if n is even.
$\frac{1}{x^{1/n}}$	$R - \{0\}$ if n is odd R^+ if n is even	$R - \{0\}$ if n is odd R^+ if n is even.
$\sin x$	R	$[-1, 1]$
$\cos x$	R	$[-1, 1]$

Apps of Derivatives — Tangents & Normals

Geometrical interpretation of a derivative

Geometrically a function is represented by a curve; its derivative at a point is represented by the slope of the tangent to the curve at that point.

$$y = f(x)$$

at (x_1, y_1) $\quad f'(x) = \left.\frac{dy}{dx}\right|_{(x_1, y_1)}$

Equation of tangent

$$y - y_1 = \left.\frac{dy}{dx}\right|_{(x_1, y_1)} \cdot (x - x_1)$$

Equation of normal

$$y - y_1 = -\left.\frac{dx}{dy}\right|_{(x_1, y_1)} (x - x_1)$$

Angle of intersection of two curves

$$\tan\theta = \frac{\left(\frac{dy}{dx}\right)_{c_1} - \left(\frac{dy}{dx}\right)_{c_2}}{1 + \left(\frac{dy}{dx}\right)_{c_1}\left(\frac{dy}{dx}\right)_{c_2}} \qquad \frac{m_1 - m_2}{1 + m_1 m_2}$$

Orthogonality of curves

$$\left(\frac{dy}{dx}\right)_{c_1} \cdot \left(\frac{dy}{dx}\right)_{c_2} = -1 \qquad m_1 m_2 = -1$$

Length of tangent $\Rightarrow \dfrac{y\sqrt{1+m^2}}{m}$

Length of normal $\Rightarrow y\sqrt{1+m^2}$

Length of sub tangent $\Rightarrow \dfrac{y}{m}$

Length of sub normal $\Rightarrow ym$.

INTEGRAL CALCULUS

Basic theorems

→ $\int k\,f(x)\,dx = k\int f(x)\,dx$. $k \to$ constant.

→ $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$,

→ $\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$

→ $\int \left(\frac{d}{dx} f(x)\right) dx = f(x)$.

Standard Integrals

(consider log as ln (base e))

→ $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

→ $\int \frac{1}{x}\,dx = \log|x| + C$

→ $\int \sin x\,dx = -\cos x + C$

→ $\int \cos x\,dx = \sin x + C$

→ $\int \sec^2 x\,dx = \tan x + C$

→ $\int \operatorname{cosec}^2 x\,dx = -\cot x + C$

→ $\int \tan x\,dx = \log|\sec x| + C$

→ $\int \cot x\,dx = \log|\sin x| + C$.

→ $\int e^x\,dx = e^x + C$

→ $\int a^x\,dx = \frac{a^x}{\ln a} + C$

→ $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$
$= -\cot^{-1}x + C$

→ $\int \frac{1}{a^2+x^2}\,dx = \tan^{-1}\frac{x}{a} + C$

→ $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

→ $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$
$= -\cos^{-1}x + C$

→ $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$

→ $\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx$
$= \ln|x + \sqrt{x^2 \pm a^2}| + C$

→ $\int \sec x \tan x\,dx$
$= \sec x + C$

→ $\int \operatorname{cosec} x \cot x\,dx$
$= -\operatorname{cosec} x + C$

→ $\int \frac{1}{|x|\sqrt{x^2-1}}\,dx$
$= \sec^{-1}x + C$
$= -\operatorname{cosec}^{-1}x + C$
($\because \operatorname{cosec}^{-1}x = \pi/2 - \sec^{-1}x$)

Note :-

* → $\int e^{x}[f(x)+f'(x)]\,dx = e^{x}f(x)+c$

→ $\int [x f'(x)+f(x)]\,dx = x f(x)+c$

Integration by partial fractions

rational function		partial fraction (form)
$\dfrac{px+q}{(x-a)(x-b)}$	$\Rightarrow$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-b)}$
$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\Rightarrow$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-a)^2}+\dfrac{C}{(x-b)}$
$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\Rightarrow$	$\dfrac{A}{(x-a)}+\dfrac{Bx+C}{x^2+bx+c}$

Extra :-

→ To solve integrals of the form $\int \dfrac{x^2+1}{x^4+\lambda x^2+1}\,dx$

(or) $\int \dfrac{x^2-1}{x^4+\lambda x^2+1}\,dx$ (or) $\int \dfrac{1}{x^4+\lambda x^2+1}\,dx$; $\lambda \in R$.

i) Divide Nr & Dr by x^2

ii) Express Dr in the form of

$$\left(x+\frac{1}{x}\right)^2 \pm k^2$$

iii) put $\left(x+\frac{1}{x}\right)=t$ (or) $\left(x-\frac{1}{x}\right)=t$.

iv) It will reduce to the form $\int \dfrac{1}{x^2+a^2}$ or $\int \dfrac{1}{x^2-a^2}$

v) Apply formula & solve.