- Secondary markets can be *exchanges* (all offers are considered simultaneously and the price is set by supply and demand) or *over the counter (OTC)* markets (traders seek counterparties in a less organized manner);
- Terminology:
  - Broker: middleman who connects sellers and buyers;
  - Dealer: trades on his own account;
  - Market maker: offers buying and selling prices to other market participants;
  - Specialist: matches buying and selling offers according to auction mechanism but may also trade on his own account;

3. Double auctions

- Most exchanges are *double auctions* where both buyers and sellers bid for securities and the *most generous offers* are selected to participate in trades;
- Exchanges can be *call auctions* or *continuous auctions*:
  (1) In a call auction, the market is cleared *once* at the end of a bidding period by matching the maximum amount of buying and selling offers;
  (2) In a continuous auction, a new offer is matched *immediately* with existing offers if possible, otherwise, the new offer is added to the *limit order book*;
- Call auctions:
  - *Selling offers*: a piecewise constant non-decreasing function $x \to s(x)$, the *supply cure*;
  - *Buying offers*: a piecewise constant non-increasing function $x \to d(x)$, the *demand curve*;
  - The *market is cleared* by matching maximum number of trades: $\bar{x} = \sup\{x|s(x) \le d(x)\}$, and the interval $[\lim_{x\uparrow\bar{x}} d(x), \lim_{x\downarrow\bar{x}} s(x)]$ consists of the market clearing prices;
  - *Market clearing* can be interpreted as *finding the social optimum:* $S(x) = \int_0^x s(z)dz$ and $D(x) = \int_0^x d(z)dz$ may be interpreted as the cost of producing x units, and the value of consuming x units;
  - *Market is cleared* by minimizing the difference $S(x) - D(x)$;
- *Convex analysis:*
  - $For\ a\ real-valued\ function\ f\ on\ an\ interval\ I\ the\ following\ are\ equivalent:$

$(a)\ f\ is\ convex,$

$$(b)\ There\ is\ a\ non-decreasing\ function\ \emptyset: I \to R\ such\ that\ f(x) = f(\bar{x}) + \int_{\bar{x}}^{x} \emptyset(z)dz\ \ for\ all\ x, \bar{x} \in I$$

$(c)\ f\ is\ differentiable\ on\ I\ except\ perhaps\ on\ a\ countable\ set, its\ derivative\ f'is\ non-decreasing\ and\ (b)\ holds\ with\ \emptyset = f';$

$\boldsymbol{Proof}\ of\ (b) \Longrightarrow (a): Let\ x_i \in I\ such\ that\ x_1 < x_2\ and\ \alpha_i > 0\ such\ that\ \alpha_1 + \alpha_2 = 1.$

$$With\ x = \alpha_1 x_1 + \alpha_2 x_2, we\ have: f(x) - f(x_1) = \int_{x_1}^{x} \emptyset(z)dz \le \int_{x_1}^{x} \emptyset(x)dz = \emptyset(x)(x - x_1)$$

$$And\ f(x_2) - f(x) = \int_{x}^{x_2} \emptyset(z)dz \ge \int_{x}^{x_2} \emptyset(x)dz = \emptyset(x)(x_2 - x)$$

$Multiply\ the\ inequalities\ by\ \alpha_1\ and\ -\alpha_2\ and\ add\ up: f(x) \le \alpha_1 f(x_1) + \alpha_2 f(x_2), thus\ f\ is\ convex.$

  - Note that $\emptyset$ in (b) is not unique: any $\emptyset$ with $f'_- \le \emptyset \le f'_+$ will do;
  - *Subgradient*: A $v \in R$ is a subgradient at $\bar{x}$ if $v \in [f'_-(\bar{x}), f'_+(\bar{x})]$; and the set of subfradients of f at $\bar{x}$ is known as the *subdifferential* of f at $\bar{x}$ and is denoted by $\partial f(\bar{x})$; Note that an $\bar{x} \in I$ minimizes f over I if and only if $0 \in \partial f(\bar{x})$;
  - Back to market clearing, an $\bar{x}$ minimizes $S(x) - D(x)$ iff $0 \in \partial(S - D)(\bar{x})$, which means:

$$0 \in [s_-(\bar{x}) - d_-(\bar{x}), s_+(\bar{x}) - d_+(\bar{x})]$$

- Limit order book (LOB): where the offers remaining after market clearing are recorded; LOB gives the marginal prices for buying or selling a given quantity at the best available prices;
- Flatter the curve s, more liquid the market;
- *Continuous auction*: market is cleared very frequently;
- Limit sell orders above the best ask-price and limit buy orders below the best bid-price *increase liquidity*; A *market order* is an order to buy/sell a given amount at the best available prices, which *reduce liquidity*;
- Price per share $\tilde{S}(x) = \frac{S(x)}{x}$, the convexity of S gives us: if $x_1 < x_2$