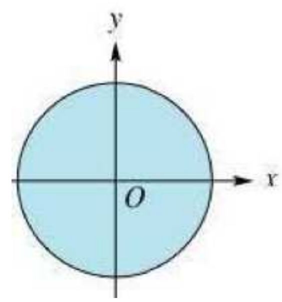
is not in the subset since the third entry does not satisfies the formula $2a-b+c=1$.

Therefore, the subset of all vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $2a-b+c=1$ is not a subspace of $\mathbf{R}^3$.

6. Consider the circle in the xy-plane centered at the origin whose equation is $x^2+y^2=1$. Let W be the set of all vectors whose tail is at the origin and whose head is a point inside or on the circle. Is W a subspace of $\mathbf{R}^2$? Explain.



Solution

Note that the radius of the circle is 1. Thus any vector $\begin{bmatrix} x \\ y \end{bmatrix}$ whose head lies outside the circle is a not vector in W. Take a vector $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in W and let a scalar r to be 2. Hence, the scalar multiple

$$r\mathbf{u} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

is not in W. Therefore, W is not a subspace.

7. Determine whether the subsets of all matrices of the following forms of $\mathbf{M}_{23}$ are subspaces.

(a) $\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix}$, where $b=a+c$

(b) $\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix}$, where $c<0$

(c) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, where $a=-3c$ and $f=3e+d$

Solution

The corresponding linear system is

$$\begin{array}{rcrcrcl} a_1 & + & a_2 & & & = & a \\ -2a_1 & + & 2a_2 & & & = & b \\ 3a_1 & - & a_2 & & & = & c \\ & & & & 3a_3 & = & d \end{array}.$$

Form the augmented matrix and row reduce it.

$$\left[\begin{array}{rrr|c} 1 & 1 & 0 & a \\ -2 & 2 & 0 & b \\ 3 & -1 & 0 & c \\ 0 & 0 & 3 & d \end{array}\right] \xrightarrow[-3R_1+R_3\to R_3]{2R_1+R_2\to R_2} \left[\begin{array}{rrr|c} 1 & 1 & 0 & a \\ 0 & 4 & 0 & b+2a \\ 0 & -4 & 0 & c-3a \\ 0 & 0 & 3 & d \end{array}\right] \xrightarrow{\frac{1}{4}R_2\to R_2}$$

$$\left[\begin{array}{rrr|c} 1 & 1 & 0 & a \\ 0 & 1 & 0 & \frac{b+2a}{4} \\ 0 & -4 & 0 & c-3a \\ 0 & 0 & 3 & d \end{array}\right] \xrightarrow{4R_2+R_3\to R_3} \left[\begin{array}{rrr|c} 1 & 1 & 0 & a \\ 0 & 1 & 0 & \frac{b+2a}{4} \\ 0 & 0 & 0 & b+c-a \\ 0 & 0 & 3 & d \end{array}\right] \xrightarrow{\frac{1}{3}R_4\to R_4}$$

$$\left[\begin{array}{rrr|c} 1 & 1 & 0 & a \\ 0 & 1 & 0 & \frac{b+2a}{4} \\ 0 & 0 & 0 & b+c-a \\ 0 & 0 & 1 & \frac{d}{3} \end{array}\right]$$

The system is consistent only when $b+c-a=0$. Therefore, the vectors do not span $\mathbf{R}_4$.

(c) $[1 \quad 1 \quad 0 \quad 0], [1 \quad 2 \quad -1 \quad 1], [0 \quad 0 \quad 1 \quad 1], [2 \quad 1 \quad 2 \quad 1]$

Solution

Let $[a \quad b \quad c \quad d]$ be an arbitrary vector in $\mathbf{R}_4$. So we have

$$a_1[1 \quad 1 \quad 0 \quad 0] + a_2[1 \quad 2 \quad -1 \quad 1] + a_3[0 \quad 0 \quad 1 \quad 1] + a_4[2 \quad 1 \quad 2 \quad 1] = [a \; b \; c \; d].$$

The corresponding linear system is

$$\begin{array}{rcrcrcrcl} a_1 & + & a_2 & & & + & 2a_4 & = & a \\ a_1 & + & 2a_2 & & & + & a_4 & = & b \\ & - & a_2 & + & a_3 & + & 2a_4 & = & c \\ & & a_2 & + & a_3 & + & a_4 & = & d \end{array}.$$

Form the augmented matrix and row reduce it.

$$\left[\begin{array}{rrrr|c} 1 & 1 & 0 & 2 & a \\ 1 & 2 & 0 & 1 & b \\ 0 & -1 & 1 & 2 & c \\ 0 & 1 & 1 & 1 & d \end{array}\right] \xrightarrow{-R_1+R_2\to R_2} \left[\begin{array}{rrrr|c} 1 & 1 & 0 & 2 & a \\ 0 & 1 & 0 & -1 & b-a \\ 0 & -1 & 1 & 2 & c \\ 0 & 1 & 1 & 1 & d \end{array}\right] \xrightarrow[-R_2+R_4\to R_4]{R_2+R_3\to R_3}$$

Solution

$$\left[\begin{array}{cccc|c} 3 & 1 & 4 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_4} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 1 & 2 & 3 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 3 & 1 & 4 & 1 & 0 \end{array}\right] \xrightarrow[\substack{-R_1+R_3\to R_3 \\ -3R_1+R_4\to R_4}]{-R_1+R_2\to R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & -2 & 0 \end{array}\right] \xrightarrow[2R_2+R_4\to R_4]{-R_2+R_1\to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right] \xrightarrow[R_3\leftrightarrow R_4]{-\frac{1}{2}R_4\to R_4}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_3+R_1\to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system has many solutions. Therefore, S is not linearly independent.

14. Which of the given vectors in $\mathbf{R}_3$ are linearly dependent? For those which are, express one vector as a linear combination of the rest.

(a) $[2 \quad -1 \quad 0], [0 \quad 3 \quad 2], [2 \quad 4 \quad 3], [3 \quad 6 \quad 6]$

Solution

Form the equation:

$$a_1[2 \quad -1 \quad 0] + a_2[0 \quad 3 \quad 2] + a_3[2 \quad 4 \quad 3] + a_4[3 \quad 6 \quad 6] = [0 \quad 0 \quad 0].$$

The corresponding linear system is

$$\begin{array}{rcrcrcrcl} 2a_1 & & & + & 2a_3 & + & 3a_4 & = & 0 \\ -a_1 & + & 3a_2 & + & 4a_3 & + & 6a_4 & = & 0 \\ & & 2a_2 & + & 3a_3 & + & 6a_4 & = & 0 \end{array}.$$

Form the augmented matrix and perform row reduction:

$$\left[\begin{array}{cccc|c} 2 & 0 & 2 & 3 & 0 \\ -1 & 3 & 4 & 6 & 0 \\ 0 & 2 & 3 & 6 & 0 \end{array}\right] \xrightarrow{-R_2\leftrightarrow R_1} \left[\begin{array}{cccc|c} 1 & -3 & -4 & -6 & 0 \\ 2 & 0 & 2 & 3 & 0 \\ 0 & 2 & 3 & 6 & 0 \end{array}\right] \xrightarrow{-2R_1+R_2\to R_2}$$

$$\left[\begin{array}{cccc|c} 1 & -3 & -4 & -6 & 0 \\ 0 & 6 & 10 & 15 & 0 \\ 0 & 2 & 3 & 6 & 0 \end{array}\right] \xrightarrow{\frac{1}{6}R_2\to R_2} \left[\begin{array}{cccc|c} 1 & -3 & -4 & -6 & 0 \\ 0 & 1 & \frac{5}{3} & \frac{5}{2} & 0 \\ 0 & 2 & 3 & 6 & 0 \end{array}\right] \xrightarrow[3R_2+R_1\to R_1]{-2R_2+R_3\to R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & \frac{5}{2} & 0 \\ 0 & 0 & -\frac{1}{3} & 1 & 0 \end{array}\right] \xrightarrow{-3R_3\to R_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & \frac{5}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 \end{array}\right] \xrightarrow[-R_3+R_1\to R_1]{-\frac{5}{3}R_3+R_2\to R_2}$$

Solution
Form the equation:

$$\begin{aligned} a_1(t^2+1)+a_2(t-2)+a_3(t+3) &= 0t^2+0t+0, \\ a_1t^2+(a_2+a_3)t+(a_1-2a_2+3a_3) &= 0t^2+0t+0. \end{aligned}$$

Equating coefficients of like powers of t, we obtain the system

$$\begin{array}{ccccccc} a_1 & & & & & = & 0 \\ & & a_2 & + & a_3 & = & 0 \\ a_1 & - & 2a_2 & + & 3a_3 & = & 0 \end{array}.$$

Next, row reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -2 & 3 & 0 \end{array}\right] \xrightarrow{-R_1+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & 3 & 0 \end{array}\right] \xrightarrow{2R_2+R_3\to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right] \xrightarrow{\frac{1}{5}R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-R_3+R_2\to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The solution is $a_1=a_2=a_3=0$. Therefore the vectors are linearly independent.

(b) $2t^2+t, t^2+3, t$

Solution
Form the equation:

$$\begin{aligned} a_1(2t^2+t)+a_2(t^2+3)+a_3t &= 0t^2+0t+0, \\ (2a_1+a_2)t^2+(a_1+a_3)t+3a_2 &= 0t^2+0t+0. \end{aligned}$$

Equating coefficients of like powers of t, we obtain the system

$$\begin{array}{ccccccc} 2a_1 & + & a_2 & & & = & 0 \\ a_1 & & & + & a_3 & = & 0 \\ & & 3a_2 & & & = & 0 \end{array}.$$

Next, row reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \xrightarrow{R_1\leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \xrightarrow{-2R_1+R_2\to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \xrightarrow{-3R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \xrightarrow{\frac{1}{6}R_3\to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[-R_3+R_1\to R_1]{2R_3+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The solution is $a_1 = a_2 = a_3 = 0$. Therefore the vectors are linearly independent.

(c) $2t^2 + t + 1, 3t^2 + t - 5, t + 13$

Solution
Form the equation:

$$\begin{aligned} a_1(2t^2+t+1) + a_2(3t^2+t-5) + a_3(t+13) &= 0t^2 + 0t + 0, \\ (2a_1+3a_2)t^2 + (a_1+a_2+a_3)t + a_1 - 5a_2 + 13a_3 &= 0t^2 + 0t + 0. \end{aligned}$$

Equating coefficients of like powers of t, we obtain the system

$$\begin{array}{ccccccc} 2a_1 & + & 3a_2 & & & = & 0 \\ a_1 & + & a_2 & + & a_3 & = & 0 \\ a_1 & - & 5a_2 & + & 13a_3 & = & 0 \end{array}.$$

Next, row reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -5 & 13 & 0 \end{array}\right] \xrightarrow{R_1\leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & -5 & 13 & 0 \end{array}\right] \xrightarrow[-R_1+R_3\to R_3]{-2R_1+R_2\to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -6 & 12 & 0 \end{array}\right] \xrightarrow[-R_2+R_1\to R_1]{6R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution is $a_3 = r, a_2 = 2r, a_1 = -3r$. Therefore the vectors is linearly dependent. Take $r = 1$, we have

$$\begin{aligned} a_1(2t^2+t+1) + a_2(3t^2+t-5) + a_3(t+13) &= 0t^2 + 0t + 0, \\ -3(2t^2+t+1) + 2(3t^2+t-5) + (t+13) &= 0t^2 + 0t + 0, \\ (t+13) &= 3(2t^2+t+1) - 2(3t^2+t-5). \end{aligned}$$

16. Which of the given vectors in $\mathbf{R}^3$ are linearly dependent? For those which are, express one vector as a linear combination of the rest.

(a) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

of A is $\{[1 \quad 3 \quad -2], [2 \quad -4 \quad 3]\}$.

32. Find a basis for the column space of A consisting of vectors that (a) are not necessarily column vectors of A; and (b) are column vectors of A.

$$A = \begin{bmatrix} 1 & -2 & 7 & 0 \\ 1 & -1 & 4 & 0 \\ 3 & 2 & -3 & 5 \\ 2 & 1 & -1 & 3 \end{bmatrix}.$$

Solution
We could perform column reduction to transform the matrix into rref. However, here we transpose matrix A and perform row reduction.

$$\begin{bmatrix} 1 & 1 & 3 & 2 \\ -2 & -1 & 2 & 1 \\ 7 & 4 & -3 & -1 \\ 0 & 0 & 5 & 3 \end{bmatrix} \xrightarrow[-7R_1+R_3\to R_3]{2R_1+R_2\to R_2} \begin{bmatrix} 1 & 1 & 3 & 2 \\ 0 & 1 & 8 & 5 \\ 0 & -3 & -24 & -15 \\ 0 & 0 & 5 & 3 \end{bmatrix} \xrightarrow[\substack{-R_2+R_1\to R_1 \\ \frac{1}{5}R_4\to R_4}]{3R_2+R_3\to R_3}$$

$$\begin{bmatrix} 1 & 0 & -5 & -3 \\ 0 & 1 & 8 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{5} \end{bmatrix} \xrightarrow[5R_4+R_1\to R_1]{-8R_4+R_2\to R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{5} \end{bmatrix} \xrightarrow{R_3\leftrightarrow R_4}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & \frac{3}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) The basis for the row space of A consisting of vectors that are not necessarily row vectors of A is $\{[1 \quad 0 \quad 0 \quad 0], [0 \quad 1 \quad 0 \quad \frac{1}{3}], [0 \quad 0 \quad 1 \quad \frac{3}{5}]\}$. Therefore the basis for column space is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{3}{5} \end{bmatrix} \right\}$$

(b) The basis for column space of A consisting of vectors that are column vectors of A is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \\ -3 \\ -1 \end{bmatrix} \right\}$$

33. Find the row and column ranks of the given matrices.