(a) $\cos\theta = \dfrac{(1)(4)+(2)(-5)}{\sqrt{1^2+2^2}+\sqrt{4^2+(-5)^2}} = \dfrac{-6}{\sqrt{5}\sqrt{41}}$

(b) $\cos\theta = \dfrac{(1)(8)+(2)(-5)}{\sqrt{1^2+2^2}+\sqrt{8^2+(-5)^2}} = \dfrac{-2}{\sqrt{5}\sqrt{89}}$

4. Determine all values of c so that $\|\mathbf{u}\| = 3$ where $\mathbf{u} = \begin{bmatrix} 2 \\ c \\ 1 \end{bmatrix}$

Solution

$$\begin{aligned} \|\mathbf{u}\| &= 3, \\ \sqrt{2^2+c^2+1^2} &= 3, \\ c^2+5 &= 9, \\ \implies c &= \pm 2. \end{aligned}$$

5. Let V be the Euclidean space $\mathbf{R}_4$ with the standard inner product. Compute $(\mathbf{u}, \mathbf{v})$.

(a) $\mathbf{u} = [1 \quad -2 \quad -3 \quad 4], \mathbf{v} = [-1 \quad 0 \quad -1 \quad -1]$

(b) $\mathbf{u} = [0 \quad -1 \quad 2 \quad 4], \mathbf{v} = [2 \quad 0 \quad -4 \quad 2]$

Solution

(a) $(\mathbf{u}, \mathbf{v}) = (1)(-1)+(-2)(0)+(-3)(-1)+(4)(-1) = -2$

(b) $(\mathbf{u}, \mathbf{v}) = (0)(2)+(-1)(0)+(2)(-4)+(4)(2) = 0$

6. Let the inner product space of continuous functions on [0,1] defined as

$$(f, g) = \int_0^1 f(t)g(t)\,dt.$$

Find (f, g) for the following:

(a) $f(t) = 3t, g(t) = -4t^2$

(b) $f(t) = t, g(t) = e^t$

Solution

(a) $(f, g) = \int_0^1 (3t)(-4t^2)\,dt = -\int_0^1 12t^3\,dt = -3t^4|_0^1 = -3.$

11. Use the Gram-Schmidt process to transform the basis $\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}4\\3\end{bmatrix}\right\}$ for the Euclidean space $\mathbf{R}^2$ into

(a) an orthogonal basis;

Solution

Let $\mathbf{u}_1=\begin{bmatrix}2\\1\end{bmatrix}, \mathbf{u}_2=\begin{bmatrix}4\\3\end{bmatrix}$. Then

(i) Let $\mathbf{v}_1=\mathbf{u}_1=\begin{bmatrix}2\\1\end{bmatrix}$.

(ii) $\mathbf{v}_2=\mathbf{u}_2-\frac{(\mathbf{u}_2,\mathbf{v}_1)}{(\mathbf{v}_1,\mathbf{v}_1)}\mathbf{v}_1=\begin{bmatrix}4\\3\end{bmatrix}-\frac{(2)(4)+(1)(3)}{(2)(2)+(1)(1)}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}4\\3\end{bmatrix}-\frac{11}{5}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}-\frac{2}{5}\\\frac{4}{5}\end{bmatrix}$.

Multiplying by 5 to clear fractions, we get

$$\mathbf{v}_2=\begin{bmatrix}-2\\4\end{bmatrix}.$$

Hence the orthogonal basis is $\left\{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}-2\\4\end{bmatrix}\right\}$

(b) an orthonormal basis.

Solution

$$\begin{aligned}\mathbf{w}_1 &= \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}=\frac{1}{\sqrt{(2)(2)+(1)(1)}}\begin{bmatrix}2\\1\end{bmatrix}=\frac{1}{\sqrt{5}}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}\frac{2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\end{bmatrix}\\ \mathbf{w}_2 &= \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}=\frac{1}{\sqrt{(-2)(-2)+(4)(4)}}\begin{bmatrix}-2\\4\end{bmatrix}=\frac{1}{\sqrt{5}}\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}\frac{-1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{bmatrix}.\end{aligned}$$

Hence the orthonormal basis is $\left\{\begin{bmatrix}\frac{2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\end{bmatrix},\begin{bmatrix}\frac{-1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{bmatrix}\right\}$.

12. Consider the Euclidean space $\mathbf{R}_4$ and let W be the susbspace that has

$$S=\{[0\quad 1\quad -2\quad 1],[1\quad 0\quad 1\quad 4]\}$$

as a basis. Use the Gram-Schmidt process to obtain an orthonormal basis for W.

Solution

Let $\mathbf{u}_1=[0\quad 1\quad -2\quad 1], \mathbf{u}_2=[1\quad 0\quad 1\quad 4]$. Then

(i) Let $\mathbf{v}_1=\mathbf{u}_1=[0\quad 1\quad -2\quad 1]$.