Tutorial Solution
Chapter 6: Linear Transformations

1. Determine whether the following functions are linear transformations?

   (a) $L : \mathbf{R}_2 \to \mathbf{R}_3$ defined by $L([u_1 \quad u_2]) = [u_1 \quad u_2 \quad u_1 + u_2]$

Solution
Let $\mathbf{u} = [u_1 \quad u_2]$ and $\mathbf{v} = [v_1 \quad v_2]$. Then

$$\begin{aligned} L(\mathbf{u}+\mathbf{v}) &= L([u_1 \quad u_2] + [v_1 \quad v_2]), \\ &= L([u_1 + v_1 \quad u_2 + v_2]), \\ &= [u_1 + v_1 \quad u_2 + v_2 \quad u_1 + u_2 + v_1 + v_2], \\ &= [u_1 \quad u_2 \quad u_1 + u_2] + [v_1 \quad v_2 \quad v_1 + v_2], \\ &= L(\mathbf{u}) + L(\mathbf{v}) \end{aligned}$$

and let c be any scalar.

$$\begin{aligned} L(c\mathbf{u}) &= = L([cu_1 \quad cu_2], \\ &= [cu_1 \quad cu_2 \quad cu_1 + cu_2], \\ &= c[u_1 \quad u_2 \quad u_1 + u_2]. \\ &= cL(\mathbf{u}). \end{aligned}$$

Therefore, L is a linear transformation.

   (b) $L : \mathbf{R}_2 \to \mathbf{R}_3$ defined by $L([u_1 \quad u_2]) = [u_1 + u_2 \quad u_2 \quad u_2 - 1]$

Solution
Let $\mathbf{u} = [u_1 \quad u_2]$ and $\mathbf{v} = [v_1 \quad v_2]$. Then

$$\begin{aligned} L(\mathbf{u}+\mathbf{v}) &= L([u_1 + v_1 \quad u_2 + v_2]), \\ &= [u_1 + v_1 + u_2 + v_2 \quad u_2 + v_2 \quad u_2 + v_2 - 1]. \end{aligned}$$

and

$$\begin{aligned} L(\mathbf{u}) + L(\mathbf{v}) &= [u_1 + u_2 \quad u_2 \quad u_2 - 1] + [v_1 + v_2 \quad v_2 \quad v_2 - 1], \\ &= [u_1 + u_2 + v_1 + v_2 \quad u_2 + v_2 \quad u_2 + v_2 - 2], \\ &\neq L(\mathbf{u}+\mathbf{v}). \end{aligned}$$

Therefore, L is not linear transformation.

2. Which of the following functions are linear transformations? [Here, $p'(t)$ denotes the derivative of $p(t)$ with respect to t.]