MATH20101

Complex Analysis

Charles Walkden

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0. Preliminaries

§0.1 Contact details

The lecturer is Dr. Charles Walkden, Room 2.241, Tel: 0161 275 5805, Email: charles.walkden@manchester.ac.uk.

My office hour is: Monday 2pm-3pm. If you want to see me at another time then please email me first to arrange a mutually convenient time.

§0.2 Course structure

§0.2.1 Lectures

There will be approximately 21 lectures in total.

The lecture notes are available on the course webpage. The course webpage is available via Blackboard or directly at `www.maths.manchester.ac.uk/~cwalkden/complex-analysis`. Please let me know of any mistakes or typos that you find in the notes.

The lectures will be recorded via the University's 'Lecture Capture' system. Remember that Lecture Capture is a useful revision tool but it is not a substitute for attending lectures.

§0.2.2 Exercises

The lecture notes also contain the exercises (at the end of each section). The exercises are an integral part of the course and you should make a serious attempt at them.

The lecture notes also contain the solutions to the exercises. I will trust you to have serious attempts at solving the exercises without looking at the solutions.

§0.2.3 Tutorials and support classes

The tutorial classes start in Week 2. There are 5 classes for this course but you only need go to one each week. You will be assigned to a class. Attendance at tutorial classes is recorded and monitored by the Teaching and Learning Office. If you go to a class other than the one you've been assigned to then you will normally be recorded as being absent.

I try to run the tutorial classes so that the majority of people get some benefit from them. Each week I will prepare a worksheet. The worksheets will normally contain exercises from the lecture notes or from past exam questions. I will often break the exercises down into easier, more manageable, subquestions; the idea is that then everyone in the class can make progress on them within the class. (If you find the material in the examples classes too easy then great!—it means that you are progressing well with the course.) You still need to work on the remaining exercises (and try past exams) in your own time!

I will not put the worksheets on the course webpage. There is nothing on the worksheets that isn't already contained in either the exercises, lecture notes or past exam papers that are already on the course webpage.

Example. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = 1$ if $z \neq 0$ and $f(0) = 0$. Then $\lim_{z\to 0} f(z) = 1$. Here $\lim_{z\to 0} f(z) \neq f(0)$.

We will be interested in functions which do behave nicely when taking limits.

Definition. Let D be a domain and let $f : D \to \mathbb{C}$ be a function. We say that f is *continuous at* $z_0 \in D$ if

$$\lim_{z\to z_0} f(z) = f(z_0).$$

We say that f is *continuous on* D if it is continuous at z_0 for all $z_0 \in D$.

Continuity obeys the same rules as in MATH20101 Real Analysis. In particular, suppose that $f, g : D \to \mathbb{C}$ are complex functions which are continuous at z_0. Then

$$f(z) + g(z),\ f(z)g(z),\ cf(z)\ (c \in \mathbb{C})$$

are all continuous at z_0, as is $f(z)/g(z)$ provided that $g(z_0) \neq 0$.

§2.4 Differentiable functions

Let us first consider how one differentiates real valued functions defined on $\mathbb{R}$. You will cover this properly in the Real Analysis course, and some of you will have seen 'differentiation from first principles' at A-level or high school. Let $(a, b) \subset \mathbb{R}$ be an open interval and let $f : (a, b) \to \mathbb{R}$ be a function. Let $x_0 \in (a, b)$. The idea is that $f'(x_0)$ is the slope of the graph of f at the point x_0. Heuristically, one takes a point x that is near x_0 and looks at the gradient of the straight line drawn between the points $(x_0, f(x_0))$ and $(x, f(x))$ on the graph of f; this is an approximation to the slope at x_0, and becomes more accurate as x approaches x_0. We then say that f is differentiable at x_0 if this limit exists, and define the derivative of f at x_0 to be the value of this limit.

Definition. Let $(a, b) \subset \mathbb{R}$ be an interval and let $x_0 \in (a, b)$. A function $f : (a, b) \to \mathbb{R}$ is *differentiable at* x_0 if

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{2.4.1}$$

exists. We call $f'(x_0)$ the *derivative of* f *at* x_0. We say that f is *differentiable* if it is differentiable at all points $x_0 \in (a, b)$.

Remark. Notice that there are two ways that x can approach x_0: x can either approach x_0 from the left or from the right. The definition of the derivative in (2.4.1) requires the limit to exist from both the left and the right and for the value of these limits to be the same.

(As an aside, one could instead look at left-handed and right-handed derivatives. For example, consider $f(x) = |x|$ defined on $\mathbb{R}$. The left-handed derivative at 0 is

$$\lim_{x\to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0-} \frac{-x}{x} = -1$$

and the right-handed derivative at 0 is

$$\lim_{x\to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0-} \frac{x}{x} = 1.$$

Thus, to calculate $\partial g/\partial x$ we treat y as a constant and differentiate with respect to x, and to calculate $\partial g/\partial y$ we treat x as a constant and differentiate with respect to y.

Theorem 2.5.1 (The Cauchy-Riemann Theorem)
Let $f : D \to \mathbb{C}$ and write $f(x+iy) = u(x,y) + iv(x,y)$. Suppose that f is differentiable at $z_0 = x_0 + iy_0$. Then

(i) the partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

exist at (x_0, y_0) and

(ii) the following relations hold

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0), \quad \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0). \tag{2.5.1}$$

Remark. The relationships in (2.5.1) are called the *Cauchy-Riemann equations*.

Proof. Recall from (2.4.2) that to calculate $f'(z_0)$ we look at points that are close to z_0 and then let these points tend to z. The trick is to calculate $f'(z_0)$ in two different ways: by looking at points that converge to z_0 horizontally, and by looking at points that converge to z_0 vertically.

Let h be real and consider $z_0 + h = (x_0 + h) + iy_0$. Then as $h \to 0$ we have $z_0 + h \to z_0$. Hence

$$\begin{aligned}
f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} \\
&= \lim_{h\to 0} \frac{u(x_0+h,y_0) + iv(x_0+h,y_0) - u(x_0,y_0) - iv(x_0,y_0)}{h} \\
&= \lim_{h\to 0} \frac{u(x_0+h,y_0) - u(x_0,y_0)}{h} + i\frac{v(x_0+h,y_0) - v(x_0,y_0)}{h} \\
&= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0).
\end{aligned} \tag{2.5.2}$$

Now take k to be real and consider $z_0 + ik = x_0 + i(y_0 + k)$. Then as $k \to 0$ we have $z_0 + ik \to z_0$. Hence

$$\begin{aligned}
f'(z_0) &= \lim_{k\to 0} \frac{f(z_0+ik) - f(z_0)}{ik} \\
&= \lim_{k\to 0} \frac{u(x_0,y_0+k) + iv(x_0,y_0+k) - u(x_0,y_0) - iv(x_0,y_0)}{ik} \\
&= \lim_{k\to 0} \frac{u(x_0,y_0+k) - u(x_0,y_0)}{ik} + i\frac{v(x_0,y_0+k) - v(x_0,y_0)}{ik} \\
&= -i\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0),
\end{aligned} \tag{2.5.3}$$

recalling that $1/i = -i$. Comparing the real and imaginary parts of (2.5.2) and (2.5.3) gives the result. $\square$

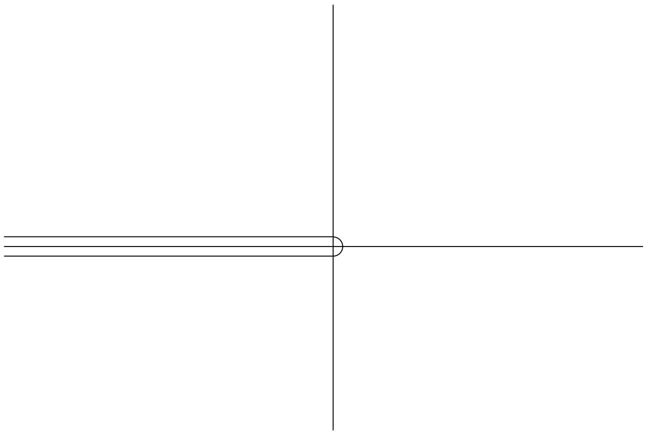

Figure 3.4.1: The cut plane: this is the complex plane with the negative real axis removed.

Definition. The complex plane with the negative real-axis (including 0) removed is called the *cut plane*. See Figure 3.4.1.

Proposition 3.4.3
The principal logarithm $\operatorname{Log} z$ *is continuous on the cut plane.*

Proof. This follows from the fact (which we shall not prove, although the proof is easy) that the principal value of the argument $\operatorname{Arg} z$ is continuous on the cut-plane. □

Having seen that the principal logarithm is continuous, we can go on to show that it is differentiable.

Proposition 3.4.4
The principal logarithm $\operatorname{Log} z$ *is holomorphic on the cut plane and*

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}.$$

Proof. Let $w = \operatorname{Log} z$. Then $z = \exp w$. Let $\operatorname{Log}(z+h) = w+k$. Then by Proposition 3.4.3 Log is continuous on the cut plane so we have that $k \to 0$ as $h \to 0$. Then

$$\begin{aligned}
\frac{d}{dz}\operatorname{Log} z &= \lim_{h\to 0}\frac{\operatorname{Log}(z+h)-\operatorname{Log}(z)}{h} \\
&= \lim_{k\to 0}\frac{(w+k)-w}{\exp(w+k)-\exp(w)} \\
&= \lim_{k\to 0}\left(\frac{\exp(w+k)-\exp(w)}{k}\right)^{-1} \\
&= \left(\frac{d}{dw}\exp(w)\right)^{-1} \\
&= \frac{1}{z}.
\end{aligned}$$

□

As an example of a path, let $z_0, z_1 \in \mathbb{C}$. Define

$$\gamma(t) = (1-t)z_0 + tz_1, \ 0 \leq t \leq 1. \tag{4.2.1}$$

Then $\gamma(0) = z_0, \gamma(1) = z_1$ and the image of γ is the straight line joining z_0 to z_1. We sometimes denote this path by $[z_0, z_1]$. See Figure 4.2.1.

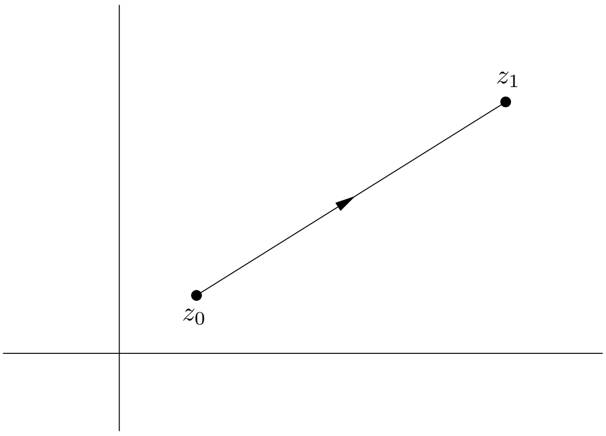

Figure 4.2.1: The path $\gamma(t) = (1-t)z_0 + tz_1$, $0 \leq t \leq 1$, describes the straight line joining z_0 to $z - 1$. We sometimes denote this path by $[z_0, z_1]$.

Definition. Let $\gamma : [a, b] \to \mathbb{C}$ be a path. If $\gamma(a) = \gamma(b)$ (i.e. if γ starts and ends at the same point) then we say that γ is a *closed path* or a *closed loop*.

Example. An important example of a closed path is given by

$$\gamma(t) = e^{it} = \cos t + i \sin t, \ 0 \leq t \leq 2\pi. \tag{4.2.2}$$

This is the path that describes the circle in $\mathbb{C}$ with centre 0 and radius 1, starting and ending at the point 1, travelling around the circle in an anticlockwise direction. See Figure 4.2.2.

Definition. A path γ is said to be *smooth* if $\gamma : [a, b] \to \mathbb{C}$ is differentiable and γ' is continuous. (By differentiable at a we mean that the one-sided derivative exists, similarly at b.)

All of the examples of paths above are smooth.

We can use integrals to define the lengths of paths:

Definition. Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth path. Then the *length of* γ is defined to be

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)| \, dt.$$

Example. It is straightforward to check from (4.2.1) that

$$\text{length}([z_0, z_1]) = |z_1 - z_0|.$$

If $\gamma(t)$ is the path given in (4.2.2) then

$$\text{length}(\gamma) = 2\pi.$$

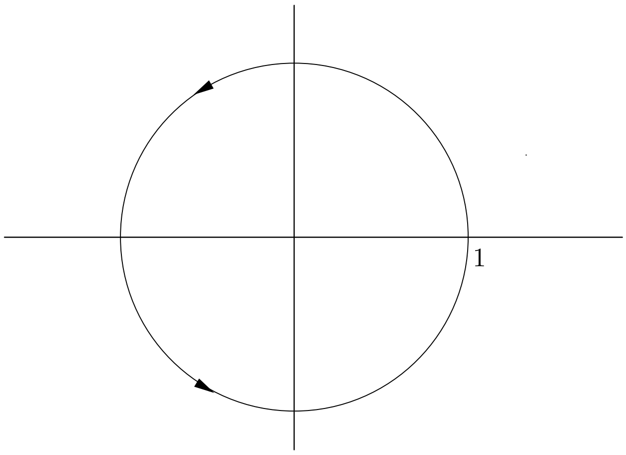

Figure 4.2.2: The circular path $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$. Note that it starts at 1 and travels anticlockwise around the unit circle.

Often we will want to integrate over a number of paths joined together. One could make the latter a path by constructing a suitable reparametrisation, but in practice this makes things complicated; in particular the joins may not be smooth. It is simpler to give a name to several smooth paths joined together.

Definition. A *contour* γ is a collection of smooth paths $\gamma_1, \ldots, \gamma_n$ where the end-point of γ_r coincides with the start point of γ_{r+1}, $1 \leq r \leq n-1$. We write

$$\gamma = \gamma_1 + \cdots + \gamma_n.$$

If the end-point of γ_n coincides with the start point of γ_1 then we call γ a *closed contour*.

Thus a contour is a path that is smooth except at finitely many places. A contour looks like a smooth path but with finitely many corners.

Example. Let $0 < \varepsilon < R$. Define

$$\begin{aligned} \gamma_1 : [\varepsilon, R] \to \mathbb{C} \qquad & \gamma_1(t) = t, \\ \gamma_2 : [0, \pi] \to \mathbb{C} \qquad & \gamma_2(t) = Re^{it}, \\ \gamma_3 : [-R, -\varepsilon] \to \mathbb{C} \qquad & \gamma_3(t) = t, \\ \gamma_4 : [-\pi, 0] \to \mathbb{C} \qquad & \gamma_4(t) = \varepsilon e^{-it}. \end{aligned}$$

Then $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ is a closed contour (see Figure 4.2.3).

Definition. The length of a contour $\gamma = \gamma_1 + \cdots + \gamma_n$ is defined to be

$$\text{length}(\gamma) = \text{length}(\gamma_1) + \cdots + \text{length}(\gamma_n).$$

Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a path that starts at $\gamma(a)$ and ends at $\gamma(b)$. Then we can consider the reverse of this path, where we start at $\gamma(b)$ and, travelling backwards along γ, end at $\gamma(a)$. More formally, we make the following definition.

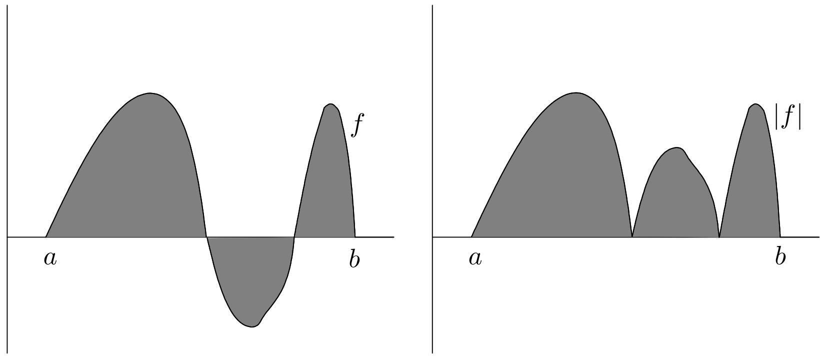

Figure 4.4.1: If $f[a,b] \to \mathbb{R}$ is negative on some subset of $[a,b]$ then the area underneath that part of the graph is negative. When f is replaced by $|f|$, this area becomes positive.

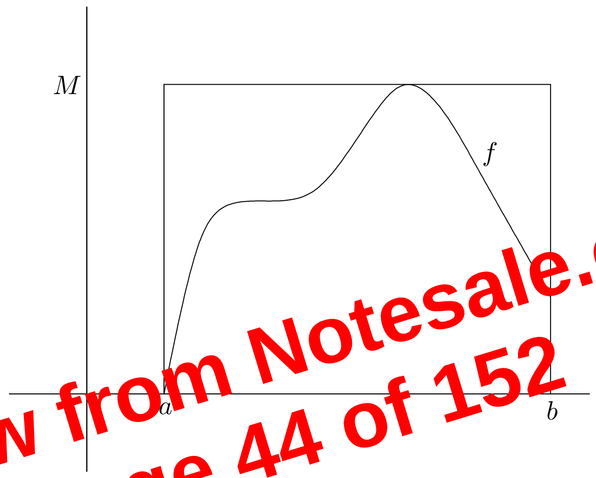

Figure 4.4.2: The graph of f is contained inside the rectangle of width $b-a$ and height M. Hence the area underneath the graph is at most $M(b-a)$.

Lemma 4.4.1
Let $u, v : [a,b] \to \mathbb{R}$ be continuous functions. Then

$$\left| \int_a^b u(t) + iv(t)\, dt \right| \leq \int_a^b |u(t) + iv(t)|\, dt. \tag{4.4.3}$$

Proof. Write

$$\int_a^b u(t) + iv(t)\, dt = X + iY.$$

Then

$$\begin{aligned} X^2 + Y^2 &= (X - iY)(X + iY) \\ &= \int_a^b (X - iY)(u(t) + iv(t))\, dt \\ &= \int_a^b Xu(t) + Yv(t)\, dt + i \int_a^b Xv(t) - Yu(t)\, dt. \end{aligned}$$

Exercises for Part 4

Exercise 4.1
Draw the following paths:

(i) $\gamma(t) = e^{-it}$, $0 \leq t \leq \pi$,

(ii) $\gamma(t) = 1 + i + 2e^{it}$, $0 \leq t \leq 2\pi$,

(iii) $\gamma(t) = t + i \cosh t$, $-1 \leq t \leq 1$,

(iv) $\gamma(t) = \cosh t + i \sinh t$, $-1 \leq t \leq 1$.

Exercise 4.2
Find the values of

$$\int_\gamma x - y + ix^2 \, dz$$

where $z = x + iy$ and γ is:

(i) the straight line joining 0 to $1 + i$;

(ii) the imaginary axis from 0 to i;

(iii) the line parallel to the real axis from i to $1 + i$.

Exercise 4.3
Let

$$\begin{aligned} \gamma_1(t) &= 2 + 2e^{it}, \ 0 \leq t \leq 2\pi, \\ \gamma_2(t) &= i + e^{-it}, \ 0 \leq t \leq \pi/2. \end{aligned}$$

Draw the paths γ_1, γ_2.

From the definition $\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)\,dt$, calculate

$$\text{(i)} \int_{\gamma_1} \frac{dz}{z-2}, \ \text{(ii)} \int_{\gamma_2} \frac{dz}{(z-i)^3}.$$

Exercise 4.4
Evaluate $\int_\gamma |z|^2 \, dz$ where γ is the circle $|z - 1| = 1$ described anticlockwise.

Exercise 4.5
For each of the following functions find an anti-derivative and calculate the integral along any smooth path from 0 to i:

(i) $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2 \sin z$;

(ii) $f : \mathbb{C} \to \mathbb{C}$, $f(z) = ze^{iz}$.

f has a Taylor series expansion given by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Furthermore, if $0 < r < R$ and $C_r(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

Remark. This version of Taylor's Theorem is false in the case of real analysis in the following sense: there are functions that are differentiable an arbitrary number of times but that are not equal to their Taylor series. For example, if

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

then f is differentiable arbitrarily many times. However, one can check (by differentiation from first principles) that $f^{(n)}(0) = 0$ for all n, so f has Taylor series 0 at 0. As $f \neq 0$ near 0, it follows that f is not equal to its Taylor series.

Definition. If, for each $z_0 \in D$, a function $f : D \to \mathbb{C}$ is equal to its Taylor series at z_0 on some open disc then we say that f is *analytic*. (It follows from Theorem 5.2.1 that all complex differentiable functions are analytic; however the example in the remark above shows that not all infinitely real-differentiable functions are analytic.)

Proof of Theorem 5.2.1. First recall that for any $w \in \mathbb{C}$ we have

$$1 + w + \cdots + w^m = \frac{1 - w^{m+1}}{1 - w}.$$

Put $w = h/(z - z_0)$. Then

$$\begin{aligned} 1 + \frac{h}{z - z_0} + \cdots + \frac{h^m}{(z - z_0)^m} &= \frac{1 - \left(\dfrac{h}{z - z_0}\right)^{m+1}}{1 - \dfrac{h}{z - z_0}} \\ &= \frac{\left(1 - \left(\dfrac{h}{z - z_0}\right)^{m+1}\right)}{z - z_0 - h} \times (z - z_0). \end{aligned}$$

Hence

$$\frac{1}{z - (z_0 + h)} = \frac{1}{z - z_0 - h} = \frac{1}{z - z_0} + \frac{h}{(z - z_0)^2} + \cdots + \frac{h^m}{(z - z_0)^{m+1}} + \frac{h^{m+1}}{(z - z_0)^{m+1}(z - z_0 - h)}.$$

Fix h such that $0 < |h| < R$ and suppose, for the moment, that $|h| < r < R$. Then Cauchy's Integral formula, together with the above identity, gives

$$\begin{aligned} &f(z_0 + h) \\ &= \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - (z_0 + h)}\, dz \end{aligned}$$

Hence $|1/p(z)| \leq 2/K^n$ for all $z \in \mathbb{C}$, so that p is a bounded holomorphic function on $\mathbb{C}$. By Liouville's Theorem (Theorem 5.3.2), this implies that p is constant, a contradiction. $\square$

Proof of Corollary 5.3.4. Let $p(z)$ be a degree n polynomial with coefficients in $\mathbb{C}$. By Theorem 5.3.3 we can find $\alpha_1 \in \mathbb{C}$ such that $p(\alpha_1) = 0$. Write $p(z) = (z - \alpha_1)q(z)$ where $q(z)$ is a degree $n-1$ polynomial with coefficients in $\mathbb{C}$. The proof then follows by induction on n. $\square$

Now Γ_R is a simple closed loop. To use Cauchy's Residue Theorem, we need to know the poles and residues of $f(z)$. Now

$$f(z) = \frac{1}{(z^2+1)(z^2+4)} = \frac{1}{(z-i)(z+i)(z-2i)(z+2i)}.$$

Hence $f(z)$ has simple poles at $z = +i, -i, +2i, -2i$. If we take $R > 2$ then the poles at $z = i, 2i$ lie inside Γ_R (note that the poles at $z = -i, -2i$ lie outside Γ_R). Now by Lemma 7.4.1,

$$\begin{aligned} \mathrm{Res}(f, i) &= \lim_{z \to i} (z-i) f(z) \\ &= \lim_{z \to i} \frac{1}{(z+i)(z-2i)(z+2i)} \\ &= \frac{1}{6i} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Res}(f, 2i) &= \lim_{z \to 2i} (z-2i) f(z) \\ &= \lim_{z \to 2i} \frac{1}{(z-i)(z+i)(z+2i)} \\ &= \frac{-1}{12i}. \end{aligned}$$

Hence by Cauchy's Residue Theorem

$$\begin{aligned} \int_{[-R,R]} f(z),\, dz + \int_{S_R} f(z)\, dz &= \int_{\Gamma_R} f(z)\, dz \\ &= 2\pi i (\mathrm{Res}(f, i) + \mathrm{Res}(f, 2i)) \\ &= 2\pi i \left(\frac{1}{6i} - \frac{1}{12i} \right) = \frac{\pi}{6}. \end{aligned}$$

If we can show that

$$\lim_{R \to \infty} \int_{S_R} f(z)\, dz = 0 \tag{7.5.6}$$

then we will have that

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\, dx = \lim_{R \to \infty} \int_{[-R,R]} f(z)\, dz = \frac{\pi}{6}.$$

To complete the calculation, we show that (7.5.6) holds. We shall use the Estimation Lemma. Let z be a point on S_R. Note that $|z| = R$. Hence

$$|(z^2+1)(z^2+4)| \geq (R^2-1)(R^2-4)$$

so that

$$\left| \frac{1}{(z^2+1)(z^2+4)} \right| \leq \frac{1}{(R^2-1)(R^2-4)}.$$

Hence, by the Estimation Lemma,

$$\begin{aligned} \left| \int_{S_R} f(z)\, dz \right| &\leq \frac{1}{(R^2-1)(R^2-4)} \mathrm{length}(S_R) \\ &= \frac{\pi R}{(R^2-1)(R^2-4)} \\ &\to 0 \end{aligned}$$

as $R \to \infty$, which is what we wanted to check.

Remark. As a general method, to evaluate

$$\int_{-R}^{R} f(x)\,dx$$

one uses the following steps:

(i) Check that $f(x)$ satisfies the hypotheses of Lemma 7.5.1.

(ii) Construct a 'D-shaped' contour Γ_R as in Figure 7.5.3.

(iii) Find the poles and residues of $f(z)$ that lie inside Γ_R when R is large.

(iv) Use Cauchy's Residue Theorem to write down $\int_{\Gamma_R} f(z)\,dz$.

(v) Split this integral into an integral over $[-R, R]$ and an integral over S_R. Use the Estimation Lemma to conclude that the integral over S_R converges to 0 as $R \to \infty$.

For a particular example, one may need to make small modifications to the above process, but the general method is normally as above.

Remark. It is very easy to lose minus signs or factors of $2\pi i$ when doing these computations. You should always check that your answer makes sense. For example, if I had missed out a factor of i in the above then I would have obtained an expression of the form

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx = \frac{\pi i}{6}.$$

This is obviously wrong: the left-hand side is a real number whereas the (incorrect) right-hand side is imaginary. Similarly, in this example the integrand on the left-hand side is a positive function, and so the integral must be positive; hence if the right-hand side is negative then there must be a mistake somewhere in the calculation.

§7.5.3 Trigonometric integrals

We can use Cauchy's Residue Theorem to calculate integrals of the form

$$\int_0^{2\pi} Q(\cos t, \sin t)\,dt \tag{7.5.7}$$

where Q is some function. (Integrands such as $\cos^4 t \sin^3 t - 7\sin t$, or $\cos t + \sin^2 t$, etc, fall into this category.)

The first step is to turn (7.5.7) into a complex integral. Set $z = e^{it}$. Then

$$\cos t = \frac{z + z^{-1}}{2}, \quad \sin t = \frac{z - z^{-1}}{2i}.$$

Also $[0, 2\pi]$ transforms into the unit circle $C_1(t) = e^{it}$, $0 \le t \le 2\pi$. Finally, note that $dz = ie^{it}\,dt$ so that

$$dt = \frac{dz}{iz}.$$

Hence

$$\int_0^{2\pi} Q(\cos t, \sin t)\,dt = \int_{C_1} Q\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}.$$

Remark. The above illustrates a more general method. For example, one can also evaluate integrals of the form

$$\int_0^{\pi} Q(\cos t, \sin t)\, dt$$

by using the substitution $z = e^{2it}$. In this case, as t varies from 0 to π then z describes the unit circle in $\mathbb{C}$ with centre 0 and radius 1 described anti-clockwise.

§7.5.4 Summation of series

Recall that $\cot \pi z = \cos \pi z / \sin \pi z$. Then $\cot \pi z$ has a pole whenever $\sin \pi z = 0$, i.e. whenever $z = n$, $n \in \mathbb{Z}$. First note that $\sin \pi z$ has a simple zero at $z = n$ (as $\sin' \pi z = \pi \cos \pi z \neq 0$ when $z = n$). Hence $\cot \pi z$ has a simple pole at $z = n$. By Lemma 7.4.1(ii) we have

$$\mathrm{Res}(\cot \pi z, n) = \frac{\cos \pi n}{\pi \cos \pi n} = \frac{1}{\pi}.$$

This suggests a method for summing infinite series of the form $\sum_{n=1}^{\infty} a_n$. Let $f(z)$ be a meromorphic function defined on $\mathbb{C}$ such that $f(n) = a_n$. Consider the function $f(z) \cot \pi z$. Then, if $f(n) \neq 0$, we have

$$\mathrm{Res}(f(z) \cot \pi z, n) = \frac{a_n}{\pi}$$

and we can use Cauchy's Residue Theorem to calculate $\sum_{n=1}^{\infty} a_n$. For example, we will show how to use this method to calculate $\sum_{n=1}^{\infty} 1/n^2$.

There are two technicalities to overcome. First of all, we need to choose a good contour to integrate around. We will want to use the Estimation Lemma along this contour, so we will need some bounds on $|f(z) \cot(\pi z)|$. Secondly, $f(z)$ may have poles of its own and these will need to be taken into account. (In the above example, to calculate $\sum_{n=1}^{\infty} 1/n^2$ we will take $f(z) = 1/z^2$, which has a pole at $z = 0$.)

Instead of choosing a D-shaped contour, here we use a square contour. Let C_N denote the square in $\mathbb{C}$ with vertices at

$$\left(N + \frac{1}{2}\right) - i\left(N + \frac{1}{2}\right), \quad \left(N + \frac{1}{2}\right) + i\left(N + \frac{1}{2}\right),$$

$$-\left(N + \frac{1}{2}\right) + i\left(N + \frac{1}{2}\right), \quad -\left(N + \frac{1}{2}\right) - i\left(N + \frac{1}{2}\right)$$

(see Figure 7.5.4). This is a square with each side having length $2N + 1$. (The factors of $1/2$ are there so that the sides of this square do not pass through the integer points on the real axis.)

Lemma 7.5.2
There is a bound, independent of N, on $\cot \pi z$ where $z \in C_N$, i.e. there exists $M > 0$ such that for all N and all $z \in C_N$, we have $|\cot \pi z| \leq M$.

Proof. Consider the square C_N. This has two horizontal sides and two vertical sides, parallel to the real and imaginary axes, respectively.

Consider first the horizontal sides. Let $z = x + iy$ be a point on one of the horizontal sides of C_N. Then $|y| \geq 1/2$. Hence

$$|\cot \pi z| \quad = \quad \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right|$$

claimed to have never learned complex analysis but could perform many real integrals using a trick called 'differentiation under the integral sign'. See `http://ocw.mit.edu/courses/mathematics/18-304-undergraduate-seminar-in-discrete-mathematics-spring-2006/projects/integratnfeynman.pdf` for an account of this, if you're interested.

(ii) (a) Here

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

so that the Cauchy-Riemann equations are satisfied.

(b) Here

$$\frac{\partial u}{\partial x} = \frac{-x^2+y^2}{(x^2+y^2)^2}, \ \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2},$$
$$\frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2}, \ \frac{\partial v}{\partial y} = \frac{-x^2+y^2}{(x^2+y^2)^2}.$$

Hence $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$ so that the Cauchy-Riemann equations hold.

(iii) When $f(z) = |z|$ we have $f(x+iy) = \sqrt{x^2+y^2}$ so that $u(x,y) = \sqrt{x^2+y^2}$, $v(x,y) = 0$. Then for $(x,y) \neq (0,0)$ we have

$$\frac{\partial u}{\partial x} = \frac{x}{(x^2+y^2)^{1/2}}, \ \frac{\partial u}{\partial y} = \frac{y}{(x^2+y^2)^{1/2}}, \ \frac{\partial v}{\partial x} = 0, \ \frac{\partial v}{\partial y} = 0.$$

If the Cauchy-Riemann equations hold then $x/(x^2+y^2)^{1/2} = 0$, $y/(x^2+y^2)^{1/2} = 0$, which imply that $x = y = 0$, which is impossible as we are assuming that $(x,y) \neq (0,0)$.

At $(x,y) = (0,0)$ we have

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{|h|}{h}$$

which does not exist. (To see this note that if $h \to 0$, $h > 0$, then $|h|/h \to 1$; however, if $h \to 0$, $h < 0$, then $|h|/h = -h/h \to -1$.)

Hence f is not differentiable anywhere.

Solution 2.4

(i) Here

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \ \frac{\partial v}{\partial y} = 3x^2 - 3y^2,$$

and

$$\frac{\partial u}{\partial y} = -6xy, \ \frac{\partial v}{\partial x} = 6xy$$

so that the Cauchy-Riemann equations hold.

(ii) Here

$$\frac{\partial u}{\partial x} = \frac{4}{(x^2+y^2)^5}(-x^5 + 10x^3y^2 - 5xy^4),$$
$$\frac{\partial v}{\partial y} = \frac{4}{(x^2+y^2)^5}(-x^5 + 10x^3y^2 - 5xy^4),$$

and

$$\frac{\partial u}{\partial y} = \frac{4}{(x^2+y^2)^5}(-5x^4 + 10x^2y^3 - y^5),$$
$$\frac{\partial v}{\partial x} = \frac{4}{(x^2+y^2)^5}(5x^4 - 10x^2y^3 + y^5)$$

10. Solutions to Part 3

Solution 3.1
Let $z_n \in \mathbb{C}$. Let

$$s_n = \sum_{k=0}^{n} z_k, \quad x_n = \sum_{k=0}^{n} \mathrm{Re}(z_k), \quad y_n = \sum_{k=0}^{n} \mathrm{Im}(z_k)$$

denote the partial sums of z_n, $\mathrm{Re}(z_n)$, $\mathrm{Im}(z_n)$, respectively. Let $s = \sum_{k=0}^{\infty} z_n$, $x = \sum_{k=0}^{\infty} \mathrm{Re}(z_k)$, $y = \sum_{k=0}^{\infty} \mathrm{Im}(z_k)$, if these exist.

Suppose that $\sum_{n=0}^{\infty} z_n$ is convergent. Let $\varepsilon > 0$. Then there exists N such that if $n \geq N$ we have $|s - s_n| < \varepsilon$. As

$$|x - x_n| \leq |s - s_n| < \varepsilon,$$

and

$$|y - y_n| \leq |s - s_n| < \varepsilon$$

(using the facts that $|\mathrm{Re}(w)| \leq |w|$ and $|\mathrm{Im}(w)| \leq |w|$ for any complex number w), provided $n \geq N$, it follows that $\sum_{k=0}^{\infty} \mathrm{Re}(z_k)$ and $\sum_{k=0}^{\infty} \mathrm{Im}(z_k)$ exist.

Conversely, suppose that $\sum_{k=0}^{\infty} \mathrm{Re}(z_k)$ and $\sum_{k=0}^{\infty} \mathrm{Im}(z_k)$ exist. Let $\varepsilon > 0$. Choose N_1 such that if $n \geq N_1$ then $|x - x_n| < \varepsilon/2$. Choose N_2 such that if $n \geq N_2$ then $|y - y_n| < \varepsilon/2$. Then if $n \geq \max\{N_1, N_2\}$ we have that

$$|z - z_n| \leq |x - x_n| + |y - y_n| < \varepsilon.$$

Hence $\sum_{k=0}^{\infty} z_k$ converges.

Solution 3.2
Recall that a formula for the radius of convergence R of $\sum a_n z^n$ is given by $1/R = \lim_{n\to\infty} |a_{n+1}|/|a_n|$ (if this limit exists).

(i) Here $a_n = 2^n/n$ so that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}}{n+1}\frac{n}{2^n} = \frac{2n}{n+1} \to 2 = \frac{1}{R}$$

as $n \to \infty$. Hence the radius of convergence is $R = 1/2$.

(ii) Here $a_n = 1/n!$ so that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0 = \frac{1}{R}$$

as $n \to \infty$. Hence the radius of convergence is $R = \infty$ and the series converges for all $z \in \mathbb{C}$.

(iii) Here $a_n = n!$ so that
$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)!}{n!} = n \to \infty = \frac{1}{R}$$
as $n \to \infty$. Hence the radius of convergence is $R = 0$ and the series converges for $z = 0$ only.

(iv) Here $a_n = n^p$ so that
$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^p}{n^p} = \left(\frac{n+1}{n}\right)^p \to 1^p = 1 = \frac{1}{R}$$
as $n \to \infty$. Hence the radius of convergence is $R = 1$.

Solution 3.3

To see that the expression in Proposition 3.2.2(i) does not converge, note that
$$\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \dfrac{2^n}{3^{n+1}} & \text{if } n \text{ is even,} \\ \dfrac{3^n}{2^{n+1}} & \text{if } n \text{ is odd.} \end{cases}$$
Hence $\lim_{n\to\infty} |a_{n+1}/a_n| = 0$ if we let $n \to \infty$ through the subsequence of even values of n but $\lim_{n\to\infty} |a_{n+1}/a_n| = \infty$ if we let $n \to \infty$ through the subsequence of odd values of n. Hence $\lim_{n\to\infty} |a_{n+1}/a_n|$ does not exist.

To see that the expression in Proposition 3.2.2(ii) does not converge, note that
$$|a_n|^{1/n} = \begin{cases} 1/2 & \text{if } n \text{ is even,} \\ 1/3 & \text{if } n \text{ is odd.} \end{cases}$$
Hence $\lim_{n\to\infty} |a_{n+1}/a_n|$ does not exist.

Note, however, that $a_n \leq 1/2^n$ for all n. Hence
$$\left|\sum_{n=0}^{\infty} a_n z^n\right| \leq \left|\sum_{n=0}^{\infty} \frac{z^n}{2^n}\right| \leq \sum_{n=0}^{\infty} \left|\frac{z}{2}\right|^n,$$
which converges provided that $|z/2| < 1$, i.e. if $|z| < 2$. Hence, by the comparison test, $\sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < 2$.

Solution 3.4

(i) We know that for $|z| < 1$
$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$
(this is the sum of a geometric progression). Hence
$$\left(\frac{1}{1-z}\right)^2 = \left(\frac{1}{1-z}\right)\left(\frac{1}{1-z}\right) = \left(\sum_{n=0}^{\infty} z^n\right)\left(\sum_{n=0}^{\infty} z^n\right).$$
Using Proposition 3.1.2 we can easily see that the coefficient of z^{n-1} in the above product is equal to n. Hence
$$\left(\frac{1}{1-z}\right)^2 = \sum_{n=1}^{\infty} n z^{n-1}.$$

and

$$\frac{\partial u}{\partial y} = \cos x \sinh y, \quad \frac{\partial v}{\partial x} = -\cos x \sinh y,$$

so that the Cauchy-Riemann equations are satisfied.

Alternatively, one could note that

$$\begin{aligned}
\cos z &= \sin\left(z + \frac{\pi}{2}\right) \\
&= \sin\left(x + \frac{\pi}{2}\right) \cosh y + i \cos\left(x + \frac{\pi}{2}\right) \sinh y \\
&= \cos x \cosh y - i \sin x \sinh y.
\end{aligned}$$

(iii) Here we have that

$$\begin{aligned}
\sinh z &= \sinh(x + iy) \\
&= \frac{1}{2}\left(e^{x+iy} - e^{-(x+iy)}\right) \\
&= \frac{1}{2}(e^x e^{iy} - e^{-x} e^{-iy}) \\
&= \frac{1}{2}((e^x \cos y - e^{-x} \cos y) + i(e^x \sin y + e^{-x} \sin y)) \\
&= \sinh x \cos y + i \cosh x \sin y.
\end{aligned}$$

Hence the real and imaginary parts of $\sinh z$ are $u(x, y) = \sinh x \cos y$ and $v(x, y) = \cosh x \sin y$, respectively.

Now

$$\frac{\partial u}{\partial x} = \cosh x \cos y, \quad \frac{\partial v}{\partial y} = \cosh x \cos y,$$

and

$$\frac{\partial u}{\partial y} = -\sinh x \sin y, \quad \frac{\partial v}{\partial x} = \sinh x \sin y,$$

so that the Cauchy-Riemann equations are satisfied.

(One can also argue, assuming the results of (i), by using the fact that $\sinh z = -i \sin iz$.)

(iv) Here we have that

$$\begin{aligned}
\cosh z &= \cosh(x + iy) \\
&= \frac{1}{2}\left(e^{x+iy} + e^{-(x+iy)}\right) \\
&= \frac{1}{2}(e^x e^{iy} + e^{-x} e^{-iy}) \\
&= \frac{1}{2}((e^x \cos y + e^{-x} \cos y) + i(e^x \sin y - e^{-x} \sin y)) \\
&= \cosh x \cos y + i \sinh x \sin y.
\end{aligned}$$

Hence the real and imaginary parts of $\cosh z$ are $u(x, y) = \cosh x \cos y$ and $v(x, y) = \sinh x \sin y$, respectively.

Now

$$\frac{\partial u}{\partial x} = \sinh x \cos y, \quad \frac{\partial v}{\partial y} = \sinh x \cos y,$$

Hence

$$
\begin{aligned}
\frac{1}{1-z+z^2} &= \frac{1-w}{1-w^3} \\
&= (1-w)(1+w^3+w^6+w^9+\cdots) \\
&= 1-w+w^3-w^4+w^6-w^7+\cdots \\
&= 1-(z-1)+(z-1)^3-(z-1)^4+(z-1)^6-(z-1)^7+\cdots
\end{aligned}
$$

provided $|z-1|<1$.

Solution 6.4
Let $f(z) = 1/(z-1)^2$.

(i) Note that $1/(z-1)^2$ is already a Laurent series centred at 1. Hence f has Laurent series

$$f(z) = \frac{1}{(z-1)^2}$$

valid on the annulus $\{z \in \mathbb{C} \mid 0 < |z-1|\}$.

(ii) Note that $f(z) = 1/(z-1)^2$ is holomorphic on the disc $\{z \in \mathbb{C} \mid |z| < 1\}$. Therefore we can apply Taylor's theorem and expand f as a power series

$$f(z) = 1 + 2z + 3z^2 + \cdots + (n+1)z^n + \cdots$$

valid on the disc $\{z \in \mathbb{C} \mid |z| < 1\}$. (To calculate the coefficients, recall that if f has Taylor series $\sum_{n=0}^{\infty} a_n z^n$ then $a_n = f^{(n)}(0)/n!$. Here we can easily compute that $f^{(n)}(z) = (-1)^n(n+1)!(z-1)^{-n-2}$ so that $f^{(n)}(0) = (n+1)!$. Hence $a_n = n+1$. Alternatively, use the method given in Exercise 3.4.) As a Taylor series is a particular case of a Laurent series, we see that f has Laurent series

$$f(z) = 1 + 2z + 3z^2 + \cdots + (n+1)z^n + \cdots$$

valid on the disc $\{z \in \mathbb{C} \mid |z| < 1\}$.

(iii) Note that

$$\frac{1}{(z-1)^2} = \frac{1}{z^2}\frac{1}{\left(1-\frac{1}{z}\right)^2}.$$

Replacing z by $1/z$ in the first part of the computation in (ii) above, we see that

$$\frac{1}{\left(1-\frac{1}{z}\right)^2} = 1 + \frac{2}{z} + \frac{3}{z^2} + \cdots + \frac{n+1}{z^n} + \cdots$$

provided $|1/z| < 1$, i.e. provided $|z| > 1$. Multiplying by $1/z^2$ we see that

$$f(z) = \frac{1}{z^2} + \frac{2}{z^3} + \frac{3}{z^4} + \cdots + \frac{n-1}{z^n} + \cdots$$

valid on the annulus $\{z \in \mathbb{C} \mid |z| > 1\}$.

Solution 6.5
Recall that a function $f(z)$ has a pole at z_0 if f is not differentiable at z_0 (indeed, it may not even be defined at z_0).

(i) The poles of $1/(z^2+1)$ occur when the denominator vanishes. Now $z^2+1 = (z-i)(z+i)$ so the denominator has zeros at $z = \pm i$ and both zeros are simple. Hence the poles of $1/(z^2+1)$ occur at $z = \pm i$ and both poles are simple.

(ii) The poles occur at the roots of the polynomial $z^4 + 16 = 0$. Let $z = re^{i\theta}$. Then we have

$$z^4 = r^4 e^{4i\theta} = -16 = 16e^{i\pi}.$$

Hence $r = 2$, $4\theta = \pi + 2k\pi$, $k \in \mathbb{Z}$. We get distinct values of z for $k = 0, 1, 2, 3$. Hence the poles are at

$$2e^{\frac{i\pi}{4} + \frac{ik\pi}{2}}, \ k = 0, 1, 2, 3,$$

or in algebraic form

$$\sqrt{2}(1+i), \ \sqrt{2}(1-i), \ \sqrt{2}(-1+i), \ \sqrt{2}(-1-i).$$

All the poles are simple.

(iii) The poles occur at the roots of $z^4 + 2z^2 + 1 = (z^2+1)^2 = (z+i)^2(z-i)^2$. The roots of this polynomial are at $z = \pm i$, each with multiplicity 2. Hence the poles occur at $z = \pm i$ and each pole is a pole of order 2.

(iv) The poles occur at the roots of $z^2 + z - 1$, i.e. at $z = (-1 \pm \sqrt{5})/2$, and both poles are simple.

Solution 6.6

(i) Since

$$\sin\frac{1}{z} = \sum_{m=0}^{\infty} (-1)^{2m+1} \frac{1}{(2m+1)! z^{2m+1}}$$

our function has infinitely many non-zero term in the principal part of its Laurent series. Hence we have an isolated essential singularity at $z = 0$.

(ii) By Exercise 5.1, the function $\sin^2 z$ has Taylor series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2} \frac{2^{2n} z^{2n}}{(2n)!}.$$

Hence

$$z^{-3}\sin^2 z = \frac{1}{z} - \frac{2^4}{2 \cdot 4!} z + \frac{2^6}{2 \cdot 6!} z^3 - \cdots$$

so that $z^{-3}\sin^2 z$ has a simple pole at $z = 0$.

(iii) Since

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

we have

$$\frac{\cos z - 1}{z^2} = \frac{-1}{2} + \frac{z^2}{4!} - \cdots$$

so that there are no terms in the principal part of the Laurent series. Hence 0 is a removable singularity.

14. Solutions to Part 7

Solution 7.1

(i) The function $f(z) = 1/z(1-z^2)$ is differentiable except when the denominator vanishes. The denominator vanishes when$z = 0, \pm 1$ and these are all simple zeros. Hence there are simple poles at $z = 0, \pm 1$. Then by Lemma 7.4.1(i) we have

$$\begin{aligned}
\mathrm{Res}(f, 0) &= \lim_{z \to 0} z \frac{1}{z(1-z^2)} = \lim_{z \to 0} \frac{1}{1-z^2} = 1; \\
\mathrm{Res}(f, 1) &= \lim_{z \to 1} (z-1) \frac{1}{z(1-z^2)} = \lim_{z \to 1} \frac{-1}{z(1+z)} = \frac{-1}{2}; \\
\mathrm{Res}(f, -1) &= \lim_{z \to -1} (z+1) \frac{1}{z(1-z^2)} = \lim_{z \to -1} \frac{1}{z(1-z)} = \frac{-1}{2}.
\end{aligned}$$

(ii) Let $f(z) = \tan z = \sin z / \cos z$. Both $\sin z$ and $\cos z$ are differentiable on $\mathbb{C}$, so $f(z)$ is differentiable except when the denominator is 0. Hence f has poles at z where $\cos z = 0$, i.e. there are poles at $(n + 1/2)\pi$, $n \in \mathbb{Z}$. These poles are simple (as $(n+1/2)\pi$ is a simple zero of $\cos z$). By Lemma 7.4.1(ii) we see that

$$\mathrm{Res}(f, (n+1/2)\pi) = \frac{\sin(n+1/2)\pi}{-\sin(n+1/2)\pi} = -1.$$

(iii) Let $f(z) = (\sin z)/z^2$. As $\sin z$ and z^2 are differentiable on $\mathbb{C}$, the poles occur when $z^2 = 0$. By considering the Taylor expansion of $\sin z$ around 0 we have that

$$\begin{aligned}
\frac{\sin z}{z^2} &= \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) \\
&= \frac{1}{z} - \frac{z}{3!} + \frac{z^2}{5!} - \cdots.
\end{aligned}$$

Hence $z = 0$ is a simple pole and $\mathrm{Res}(f, 0) = 1$.

(iv) Let $f(z) = z/(1+z^4)$. This has poles when the denominator vanishes, i.e. when $z^4 = -1$. To solve this equation, we work in polar coordinates. Let $z = re^{i\theta}$. Then $z^4 = -1$ implies that $r^4 e^{4i\theta} = e^{i\pi}$. Hence $r = 1$ and $4\theta = \pi + 2k\pi$. Hence the four quartic roots of -1 are:

$$e^{i\pi/4}, \ e^{3i\pi/4}, \ e^{-i\pi/4}, e^{-3i\pi/4}.$$

These are all simple zeros of $z^4 = -1$. Hence by Lemma 7.4.1(ii) we have that $\mathrm{Res}(f, z_0) = z_0/4z_0^3 = 1/4z_0^2$ so that

$$\begin{aligned}
\mathrm{Res}(f, e^{i\pi/4}) &= \frac{1}{4e^{i\pi/2}} = \frac{1}{4i} = \frac{-i}{4} \\
\mathrm{Res}(f, e^{3i\pi/4}) &= \frac{1}{4e^{3\pi/2}} = \frac{1}{-4i} = \frac{i}{4} \\
\mathrm{Res}(f, e^{-i\pi/4}) &= \frac{1}{4e^{-i\pi/2}} = \frac{1}{-4i} = \frac{i}{4} \\
\mathrm{Res}(f, e^{-3i\pi/4}) &= \frac{1}{4e^{-3i\pi/2}} = \frac{1}{4i} = \frac{-i}{4}.
\end{aligned}$$

(ii) Here we have the same integrand as in (i) but integrated over the smaller circle $C_{5/2}$. This time only the pole $z = 2$ lies inside $C_{5/2}$. Hence

$$\int_{C_{5/2}} \frac{1}{z^2 - 5z + 6}\,dz = 2\pi i\,\mathrm{Res}(f, 2) = -2\pi i.$$

(iii) Let f denote the integrand. Note that

$$\frac{e^{az}}{1+z^2} = \frac{e^{az}}{(z+i)(z-i)}.$$

Hence f has simple poles at $z = \pm i$. Now

$$\begin{aligned}
\mathrm{Res}(f, i) &= \lim_{z\to i} \frac{(z-i)e^{az}}{(z-i)(z+i)} = \lim_{z\to i} \frac{e^{az}}{z+i} = \frac{e^{ia}}{2i} \\
\mathrm{Res}(f, -i) &= \lim_{z\to -i} \frac{(z+i)e^{az}}{(z-i)(z+i)} = \lim_{z\to -i} \frac{e^{az}}{z-i} = -\frac{e^{-ia}}{2i}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{C_2} \frac{e^{az}}{1+z^2}\,dz &= 2\pi i\left(\mathrm{Res}(f, i) + \mathrm{Res}(f, -i)\right) \\
&= 2\pi i\left(\frac{e^{ia}}{2i} - \frac{e^{-ia}}{2i}\right) \\
&= 2\pi i \sin a.
\end{aligned}$$

Solution 7.3

Suppose f has Laurent series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ valid on the annulus $\{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$. By Theorem 6.2.1 the coefficients a_n are given by

$$a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

where C_r is a circular path described anticlockwise centred at z_0 and with radius r, where r is chosen such that $R_1 < r < R_2$.

(i) We calculate that Laurent series of $f(z) = 1/z(z-1)$ valid on the annulus $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Here $z_0 = 0$. Choose $r \in (0, 1)$. We have that

$$a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z^{n+1}}\,dz = \frac{1}{2\pi i}\int_{C_r} \frac{1}{z^{n+2}(z-1)}\,dz$$

where C_r is the circular path with centre 0 and radius $r \in (0, 1)$, described once anticlockwise.

It is straightforward to locate the singularities of the integrand. For all $n \in \mathbb{Z}$ the integrand has a simple pole at 1. When $n \geq -1$, the integrand also has a pole of order $n + 2$ at 0.

For $n = -2, -3, \ldots$ the integrand has no poles inside C_r when $r < 1$. Hence, by Cauchy's Residue Theorem, $a_n = 0$ for $n = -2, -3, \ldots$. For $n \geq -1$, the pole at 0 lies

This has poles where the denominator vanishes, i.e. at $z = \pm ia, \pm ib$, and all of these poles are simple. If R is taken to be larger than b then the poles inside Γ_R occur at $z = ia, ib$. We can calculate

$$\begin{aligned}\mathrm{Res}(f, ia) &= \lim_{z \to ia} \frac{(z - ia)ze^{iz}}{(z - ia)(z + ia)(z^2 + b^2)} \\ &= \lim_{z \to ia} \frac{ze^{iz}}{(z + ia)(z^2 + b^2)} \\ &= \frac{iae^{-a}}{2ia(b^2 - a^2)} \\ &= \frac{e^{-a}}{2(b^2 - a^2)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mathrm{Res}(f, ib) &= \lim_{z \to ib} \frac{(z - ib)ze^{iz}}{(z - ib)(z + ib)(z^2 + a^2)} \\ &= \lim_{z \to ib} \frac{ze^{iz}}{(z + ib)(z^2 + a^2)} \\ &= \frac{ibe^{-b}}{2ib(-b^2 + a^2)} \\ &= \frac{-e^{-b}}{2(b^2 - a^2)}.\end{aligned}$$

Hence

$$\begin{aligned}\int_{[-R,R]} f\,dz + \int_{S_R} f\,dz &= \int_{\Gamma_R} f\,dz \\ &= 2\pi i\,(\mathrm{Res}(f, ia) + Res(f, ib)) \\ &= \frac{2\pi i}{2(b^2 - a^2)}(e^{-a} - e^{-b})\end{aligned}$$

provided that $R > b$.

Now if z is a point on S_R then $|z| > R$. Hence

$$|(z^2 + a^2)(z^2 + b^2)| \geq (|z|^2 - a^2)(|z|^2 - b^2) = (R^2 - a^2)(R^2 - b^2).$$

Also, writing $z = x + iy$ so that $0 \leq y \leq R$, we have that $|e^{iz}| = |e^{i(x+iy)}| = |e^{-y+ix}| = |e^{-y}| \leq 1$. Hence

$$|f(z)| \leq \frac{R}{(R^2 - a^2)(R^2 - b^2)}.$$

By the Estimation Lemma,

$$\int_{S_R} f(z)\,dz \leq \frac{R}{(R^2 - a^2)(R^2 - b^2)}\,\mathrm{length}(S_R) = \frac{\pi R^2}{(R^2 - a^2)(R^2 - b^2)}$$

which tends to zero as $R \to \infty$. Hence

$$\int_{[-R,R]} f\,dz = \frac{\pi i}{(b^2 - a^2)}(e^{-a} - e^{-b}).$$

By taking the imaginary part, we see that

$$\int_{-R}^{R} \frac{x \sin x}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{(b^2-a^2)}(e^{-a}-e^{-b}).$$

(As a check to see if we have made a mistake, note that the real part is zero. Hence

$$\int_{-\infty}^{\infty} \frac{x \cos x}{(x^2+a^2)(x^2+b^2)}\, dx = 0.$$

This is obvious as the integrand is an even function, and so must integrate (from $-\infty$ to ∞) to zero.)

Solution 7.8
Denote by C the unit circle $C(t) = e^{it}$, $0 \le t \le 2\pi$.

(i) Substitute $z = e^{it}$. Then $dz = ie^{it}\, dt = iz\, dt$ so that $dt = dz/iz$ and $[0, 2\pi]$ transforms to C. Also, $\cos t = (z + z^{-1})/2$. Hence

$$\int_0^{2\pi} 2\cos^3 t + 3\cos^2 t\, dt = \int_C \left(\frac{(z+z^{-1})^3}{4} + \frac{3(z+z^{-1})^2}{4} \right) \frac{dz}{iz}.$$

Now

$$\begin{aligned} \frac{(z+z^{-1})^3}{4} &= \frac{z^3+3z+3z^{-1}+z^{-3}}{4}, \\ \frac{(z+z^{-1})^2}{4} &= \frac{3z^2+6+3z^{-1}}{4}. \end{aligned}$$

Hence

$$\begin{aligned} &\int_0^{2\pi} 2\cos^3 t + 3\cos^2 t\, dt \\ &= \int_C \frac{1}{i} \left(\frac{1}{4z^4} + \frac{3}{4z^3} + \frac{3}{4z^2} + \frac{3}{2z} + \frac{3}{4} + \frac{2z}{4} + \frac{z^2}{4} \right) dz. \end{aligned}$$

Now the integrand has a pole of order 4 at $z = 0$, which is inside C, and no other poles. We can immediately read off the residue at $z = 0$ as the coefficient of $1/z$, namely $3/2i$. Hence by the Residue Theorem

$$\int_0^{2\pi} 2\cos^3 t + 3\cos^2 t\, dt = 2\pi i \frac{3}{2i} = 3\pi.$$

(ii) As before, substitute $z = e^{it}$. Then $dt = dz/iz$, $\cos t = (z + z^{-1})/2$ and $[0, 2\pi]$ transforms to C. Hence

$$\int_0^{2\pi} \frac{1}{1+\cos^2 t}\, dt = \int_C \frac{1}{1+(z+z^{-1})^2/4} \frac{dz}{iz} = \frac{1}{i} \int_C \frac{4z}{z^4+6z^2+1}\, dz$$

Let

$$f(z) = \frac{4z}{z^4+6z^2+1}.$$