7. Multiplication by the identity (or unit) matrix
   Given matrix A, there exists a matrix I such that $AI = A$

   E.g.

$$\begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix}_{2\times2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times2} = \begin{pmatrix} 1 & 3 \\ 5 & 2 \end{pmatrix}_{2\times2}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 5 & 2 \\ -1 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 5 & 2 \\ -1 & 8 \end{pmatrix}$$

Determinant

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$detA = |A| = ad - bc$$

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$detB = |B| = a(ei - hf) - b(di - gf) + c(dh - ge)$$

E.g.

$$A = \begin{pmatrix} 6 & 1 \\ 3 & 2 \end{pmatrix} \qquad |A| = 12 - 3 = 9$$

$$B = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \qquad |B| = \frac{3}{4} + \frac{1}{4} = 1$$

$$C = \begin{pmatrix} -4 & -2 \\ 6 & 3 \end{pmatrix} \qquad |c| = -12 - (-12) = 0$$

Can be positive, negative or zero

Minor:

M_{ij} of an element A_{ij} of matrix A is the determinant of the submatrix obtained by deleting the i^{th} row and j^{th} column

Co-factor:

C_{ij} of element A_{ij} is $(-1)^{i+j}M_{ij}$ and determines whether the element of the determinant is positive or negative

E.g.

$$A = \begin{pmatrix} 3 & -2 & 6 \\ 4 & 2 & -1 \\ 0 & 5 & 3 \end{pmatrix}$$

2. $B = \begin{pmatrix} 3 & -2 & -6 \\ 4 & 2 & -1 \\ 0 & 5 & -3 \end{pmatrix}$ $\qquad |B| = B_{11}C_{11} + B_{21}C_{21} + B_{31}C_{31}$

$$C_{11} = \begin{vmatrix} 2 & -1 \\ 5 & -3 \end{vmatrix} = -1 \qquad C_{21} = \begin{vmatrix} -2 & 6 \\ 5 & -3 \end{vmatrix} = 36 \qquad B_{31} = 0$$

$$|B| = -3 - 144 = 147$$

Physical application

1. Simultaneous equation:

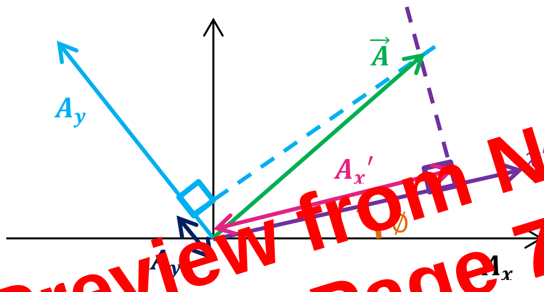
$$ax + by = e$$
$$cx + dy = f$$

Cast in matrix form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$$

Matrix form: $\qquad Ax = B \qquad X = A^{-1}B$

2. Rotation:



$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} = A_x{}'\hat{\imath}' + A_y{}'\hat{\jmath}'$$

$$\begin{pmatrix} A_x{}' \\ A_y{}' \end{pmatrix} = \begin{pmatrix} \cos\emptyset & \sin\emptyset \\ -\sin\emptyset & \cos\emptyset \end{pmatrix}\begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

$$A'_x = A_x\cos\emptyset - A_y\sin\emptyset \qquad A'_y = A_y\sin\emptyset + A_y\cos\emptyset$$

Notes

1. $detA = 0$ $\qquad$ no solution for $X = A^{-1}B$

2. $detA \neq 0$
   a. $A^{-1} = 0$ → $X = 0$ $\qquad$ (unlikely to happen)

   b. Solution can be: $X = E$, sum might$= 0$

   i.e. $\begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$ $\qquad f_1p + f_2q = 0$

   c. If the equations give the same line there is an infinite number of solutions:

   i.e. $\begin{pmatrix} -1 & 6 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad |A| = 0 \qquad y = \frac{x}{2}$

1. $I = \int xe^{-x}dx$

$$u = x \qquad v' = e^{-x} \qquad v = -e^{-x} \qquad u' = 1$$

$$\int uv'dx = uv - \int u'vdx = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x} + c$$

2. $I = \int \sin x \cos x \; dx$

$$u = \sin x \qquad v' = \cos x \qquad v = \sin x \qquad u' = \cos x$$

$$\int uv'dx = uv - \int u'vdx = \sin^2 x - \int \sin x \cos x \; dx$$

$$2I = \sin^2 x + c$$

$$I = \frac{1}{2}(\sin^2 x + c)$$

3. $I = x^2 \int \cos x \; dx$

$$u = x^2 \qquad v' = \cos x \qquad v = \sin x \qquad u' = 2x$$

$$I = x^2 \sin x - \int 2x \sin x$$

Let $J = \int x \sin x$

$$u = x \qquad v' = \sin x \qquad v = -\cos x \qquad u' = 1$$

$$J = -x\cos x - \int -\cos x \, dx = -x\cos x + \sin x + c$$

$$I = x^2 \sin x + 2x\cos x - 2\sin x + c$$

4.

$$\int_a^b \frac{2}{x}dx = 2\ln x|_a^b = 2\ln\frac{b}{a}$$

Integrand doesn't fall off enough to produce finite result as $b \to \infty$, integral diverges as $a \to 0$

5.

$$\int \sqrt{4x-1}\, dx = \int (4x-1)^{\frac{1}{2}}\, dx = \frac{(4x-1)^{\frac{3}{2}}}{6} + c$$

6.

$$\int \cos^2 x - \sin^2 x \; dx = \int \cos 2x \; dx = \frac{\sin 2x}{2} + c$$

7.

Implicit differentiation

$$g(y) = h(x)$$

By chain rule:

$$\frac{dg}{dy} \times \frac{dy}{dx} = \frac{dh}{dx}$$

$$\frac{dy}{dx} = \frac{dh/dx}{dg/dy}$$

$$y = x^{\frac{1}{n}} \quad \rightarrow \quad y^n = x$$

$$h = x \quad \rightarrow \quad h' = 1 \qquad\qquad g = y^n \quad \rightarrow \quad g' = ny^{n-1}$$

E.g.

$$y' = \frac{1}{ny^{n-1}} = \frac{1}{nx^{\frac{n-1}{n}}}$$

Power rule:

$$y' = \frac{1}{n}x^{\frac{1}{n}-1} = \frac{1}{n}x^{\frac{1-n}{n}} = \frac{1}{n}\frac{1}{x^{\frac{1-n}{n}}}$$

Logarithmic functions

$$y(x) = \ln x \quad \rightarrow \quad x = e^y$$

Implicit differentiation:

$$y' = \frac{h'}{g'} = \frac{1}{e^y} = \frac{1}{x}$$

Parametric differentiation

$$y = u(t) \qquad\qquad x = v(t)$$

$$y' = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}}$$

E.g.

$$x = r\cos(\omega t) \qquad\qquad y = r\sin(\omega t)$$

$$y' = \frac{r\cos(\omega t)}{-r\sin(\omega t)} = -\frac{1}{\tan(\omega t)} = -\cot(\omega t)$$

$$mean: \quad \mu_x = \langle x \rangle = \int_{-\infty}^{\infty} x\, P(x) dx$$

$$variance: \quad var(x) = {\sigma_x}^2$$

Mode = median = mean

$$P(x_1 < x < x_2) = \int_{x_1}^{x_2} P(x) dx = \frac{1}{\sigma_x \sqrt{2\pi}} \int_{x_1}^{x_2} exp\left(-\frac{(x-\mu_x)^2}{2{\sigma_x}^2}\right) dx$$

Sub $y = \frac{x-\mu_x}{\sqrt{2}\sigma_x}$

$$P(x_1 < x < x_2) = \frac{1}{\sqrt{\pi}} \int_{y_1}^{y_2} e^{-y^2}\, dy = \frac{1}{\sqrt{\pi}} \left[\int_0^{y_2} e^{-y^2}\, dy - \int_0^{y_1} e^{-y^2}\, dy\right]$$

Error function

$$\text{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^{y} e^{-y^2}\, dy$$

$$P = \frac{1}{2}[\text{erf}(y_2) - \text{erf}(y_1)]$$

$$Arg(z_2) = \theta_0 + \pi = \frac{5\pi}{4}$$

$$z_2 = \sqrt{2}\left(\cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4}\right)$$

$z_3 = -1 + j$ $\qquad |z_3| = \sqrt{2}$ $\qquad \theta_0 = \tan^{-1}(-1) = -\frac{\pi}{4}$

1st quadrant → k = 1

$$Arg(z_2) = \theta_0 + \pi = \frac{3\pi}{4}$$

$$z_3 = \sqrt{2}\left(\cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}\right)$$

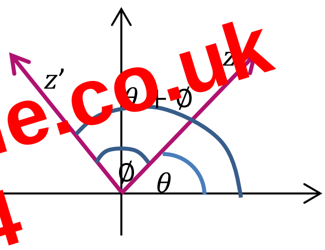
Multiplication in polar representation

$$z = p(\cos\theta - j\sin\theta)$$

$$p = |z|$$

$$n = \cos\emptyset + j\sin\emptyset$$

$$|n| = 1$$



$$z' = z \times n = p(\cos\theta - j\sin\theta) \times (\cos\emptyset + j\sin\emptyset)$$
$$= p[(\cos\theta\cos\emptyset - \sin\theta\sin\emptyset) + j(\sin\theta\cos\emptyset + \sin\emptyset\cos\theta)]$$
$$= p[\cos(\theta + \emptyset) + j\sin(\theta + \emptyset)]$$

Equivalent (more complicated) representation of rotation:

$z = x + jy$ $\qquad z' = x' + jy'$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\emptyset & -\sin\emptyset \\ \sin\emptyset & \cos\emptyset \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

E.g. $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad z_v = 1 + j$

1. Rotate by $\frac{\pi}{4}$:

$$z'_v = (1+j)\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right) = 0 + j$$

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

2. Rotate by $\frac{\pi}{2}$:

$$z'_v = (1+j)\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right) = (1+j)j$$

Rotation by $\frac{\pi}{2}$ → multiply by j

Rotation by $-\frac{\pi}{2}$ → divide by j

$$z_1 = p_1(\cos\theta_1 - j\sin\theta_1) \qquad z_2 = p_2(\cos\theta_2 - j\sin\theta_2)$$

$$z_1 \times z_2 = p_1p_2(\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2))$$

$$\frac{z_1}{z_2} = \frac{p_1}{p_2}(\cos(\theta_1-\theta_2) + j\sin(\theta_1-\theta_2))$$

Euler's Theorem

$$e^{x_1}e^{x_2} = e^{(x_1+x_2)} \qquad \frac{e^{x_1}}{e^{x_2}} = e^{(x_1-x_2)}$$

We can identify: $e^{j\theta} = \cos\theta + j\sin\theta$

Introduce exponential notation:

$$z_1 = p_1e^{j\theta_1} \qquad z_2 = p_2e^{j\theta_2}$$

Multiplication and division in exponential representation:

$$z_1z_2 = p_1p_2e^{j\theta_1}e^{j\theta_2} = p_1p_2e^{j(\theta_1+\theta_2)} = p_1p_2[\cos(\theta_1+\theta_2) + j\sin(\theta_1-\theta_2)]$$

$$\frac{z_1}{z_2} = \frac{p_1}{p_2}e^{j(\theta_1-\theta_2)}$$

E.g. $(1+j)^{10}$ $|z| = 2$

$$1 + j = \sqrt{2}e^{j\frac{\pi}{4}}$$

$$(1+j)^{10} = \left(\sqrt{2}\right)^{10}e^{j\cdot 10\frac{\pi}{4}} = 32e^{j\frac{\pi}{2}} = 32\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right) = 32j$$

Complex roots

$$z = pe^{j\theta} = pe^{j(\theta+2k\pi)}$$

θ is the principle argument, k is an integer

$$\sqrt[n]{z} = z^{\frac{1}{n}} = p^{\frac{1}{n}}\, exp\left(\frac{j(\theta+2k\pi)}{n}\right)$$

For k = 0, 1, 2 … n-1 (there are n different roots)

All other values of k are repeats

E.g.

1. $\sqrt{1} = \sqrt{e^{j\cdot 2k\pi}} = e^{jk\pi}$

   2 roots:

$$\Delta f \approx \vec{\nabla} f\big|_{x_0,y_0} \cdot \Delta\vec{R} + \frac{1}{2}\Delta\vec{R} \cdot H(f)\Delta\vec{R}$$

Taylor expansion

$H(f)$ = Hessian matrix

First term component:

$$\vec{\nabla} f \Delta\vec{R} = \frac{\partial f}{\partial x}\bigg|_{x_0,y_0} \Delta x + \frac{\partial f}{\partial y}\bigg|_{x_0,y_0} \Delta y \leq 0$$

≤ 0 for generic displacement

We need:

$$\frac{\partial f}{\partial x}\bigg|_{x_0,y_0} = \frac{\partial f}{\partial y}\bigg|_{x_0,y_0} \equiv 0$$

OBVIOUS: on tangent to plane parallel to x-y plane

Second component

$$\Delta\vec{R} \cdot H(f)\Delta\vec{R} \leq 0$$

Scalar product

$H(f)$ has 2 eigenvalues (λ_1 and λ_2) and 2 eigenvectors ($\vec{v}_1$ and $\vec{v}_2$)

Write: $\Delta\vec{R} = h\vec{v}_1 + k\vec{v}_2$

And say: $|\vec{v}_1| + |\vec{v}_2| = 1$

$$[h\vec{v}_1 + k\vec{v}_2] \cdot H(f)[h\vec{v}_1 + k\vec{v}_2] = [h\vec{v}_1 + k\vec{v}_2] \cdot [h\lambda_1\vec{v}_1 + k\lambda_2\vec{v}_2] = h^2\lambda_1 + k^2\lambda_2 \leq 0$$

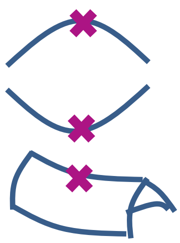
$$(\vec{v}_1 \cdot \vec{v}_2 = 0)$$

We need: $\lambda_1 \leq 0$ $\lambda_2 \leq 0$

Conditions for extrema

1) Study $\vec{\nabla} f = 0$ (gradient) and use this to figure out the extrema
2) Build $H(f)$ calculated in extrema
3) Calculate eigenvalues

	λ_1	λ_2
Maxima	<0	<0
Minima	>0	>0
Saddle	>0	<0
	<0	>0



$$\frac{\partial \emptyset}{\partial x} = 0 \qquad \frac{\partial \emptyset}{\partial y} = 0 \qquad \frac{\partial \emptyset}{\partial \mu} = g(x, y) = 0$$

E.g.

1. $f(x, y) = 3x^2 + 2y^2$ subject to the constraint $x + y = 1$

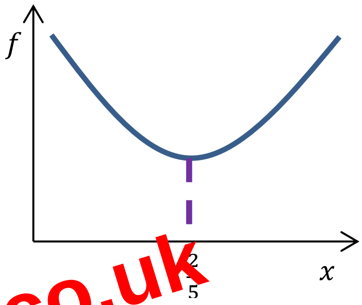
$$\emptyset(x, y, \mu) = 3x^2 + 2y^2 - \mu(x + y - 1)$$

Find the gradient and set to zero:

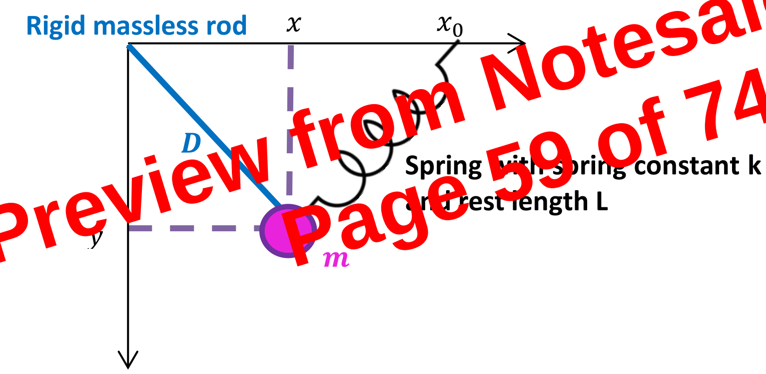
$$\frac{\partial \emptyset}{\partial x} = 6x - \mu = 0$$
$$\frac{\partial \emptyset}{\partial y} = 4y - \mu = 0$$
$$\frac{\partial \emptyset}{\partial \mu} = x + y - 1 = 0$$

Solve for x, y, μ:

$$\mu = \frac{12}{5} \qquad x = \frac{2}{5} \qquad y = \frac{3}{5}$$



2. Find the equilibrium position



Potential energy:

$$U(x, y) = -mgy + \frac{1}{2}\left[\sqrt{(x_0 - x)^2 + y^2} - L\right]^2$$

$$\text{Spring length} = \sqrt{(x_0 - x)^2 + y^2}$$

Find the minimum of U with constant $x^2 + y^2 = D^2$:

$$\emptyset(x, y, \mu) = U(x, y) - \mu(x^2 + y^2 - D^2)$$

$$\frac{\partial \emptyset}{\partial x} = 0 \qquad \frac{\partial \emptyset}{\partial y} = 0 \qquad \frac{\partial \emptyset}{\partial \mu} = x^2 + y^2 - D^2 = 0$$

Additional knowledge:

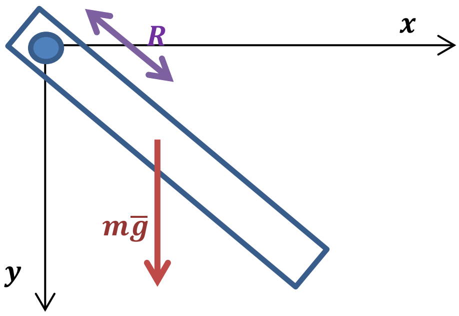
$$t = 0 \qquad \rightarrow \qquad Q(t) = 0$$

$$Q(0) = A + \varepsilon_b C = 0$$

$$A = -\varepsilon_b C$$

$$Q(t) = \varepsilon_b C \left[1 - e^{\frac{-t}{\tau}}\right]$$

2. Equation of motion



Moment of inertia:

$$I_p = \frac{d\bar{\omega}}{dt} = \bar{T}$$

$\bar{T}$ = toque

$$\bar{\omega} = \omega \hat{k}$$

$$\bar{T} = -mgR\sin\theta\,\hat{k} = \bar{R} \times m\bar{g}$$

$$I_p = I_C + mR^2$$

$$I_p \frac{d^2\theta}{dt^2}\hat{k} = -mgR\sin\theta\,\hat{k}$$

2nd order differential equation, non-linear

For a small oscillation, i.e.: $\theta \ll 1$

$$\sin\theta \approx \theta$$

First order Taylor expansion

$$I_p \frac{d^2\theta}{dt^2} = -mgR\theta$$

Solve: $\theta(t) = \sin(\omega t)$

$$\frac{d\theta}{dt} = \omega\cos(\omega t) \qquad \frac{d^2\theta}{dt^2} = -\omega^2\sin(\omega t) = \frac{d\bar{\omega}}{dt}$$

$$-I_p\omega^2\sin(\omega t) = -mgR\sin(\omega t)$$

$$\omega^2 = \frac{mgR}{I_p}$$

There is not only one solution:

$$\theta(t) = A\sin(\omega t)$$

$$\theta(t) = B\cos(\omega t)$$

Most general solution:

$$\theta(t) = A\sin(\omega t) + B\cos(\omega t)$$

To fix A and B use additional knowledge:

2nd order differential linear equation with constant coefficient:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x)$$

If $g(x) = 0$ →homogeneous differential equation

If $g(x) \neq 0$ →non-homogeneous differential equation

Differential operator

$$g(x) = \frac{dy}{dx} = \left[\frac{d}{dx}\right] y(x) = L\, y(x)$$

Define L:

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c$$

Linearity

When the differential operator applied to the sum of functions is equal to the sum of the differential operators applied to each function

Say 2 function: $y_1(x), y_2(x)$ and 2 coefficients: λ_1, λ_2

$$L[\lambda_1 y_1(x) + \lambda_2 y_2(x)] = \lambda_1\left[a\frac{d^2y_1}{dx^2} + b\frac{dy_1}{dx} + cy_1\right] + \lambda_2\left[a\frac{d^2y_2}{dx^2} + b\frac{dy_2}{dx} + cy_2\right]$$
$$= \lambda_1 L y_1 + \lambda_2 L y_2$$

Homogeneous linear 2nd order differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

If $y_1(x)$ is a solution and $y_2(x)$ is a solution also then $y(x) = \lambda_1 y_1 + \lambda_2 y_2$ is also a solution

2 linearly independent solutions → any other solution can be obtained from their sum with coefficients

Linearly dependent and independent functions

$$\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 = 0$$

If it is satisfied only for $\lambda_1 = \lambda_2 = \lambda_3 = 0$, y_1, y_2, y_3 are said to be independent

If $\lambda_3 \neq 0$:

$$y_3(x) = -\frac{\lambda_1}{\lambda_3} y_1(x) - \frac{\lambda_2}{\lambda_3} y_2(x)$$

y_1, y_2, y_3 are said to be linearly dependent (1 function can be expressed as a combination of the other 2)

E.g.

1) $\lambda_1 \cos(\theta) + \lambda_2 \sin(\theta) = 0$ → $\lambda_1 = \lambda_2 = 0$

1) Find just 1 solution (particular solution, $y_p(x)$)

2) Write associated homogeneous equation: $(f(x) = 0)$

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

3) Find the general solution/complimentary function:

$$y_c(x) = \lambda_1 y_1(x) + \lambda_2 y_2(x)$$

4) General solution of the non-homogeneous differential equation is:

$$y(x) = y_p(x) + y_c(x) = y_p(x) + \lambda_1 y_1(x) + \lambda_2 y_2(x)$$

Particular solution + general solution

Then we have:

$$y(x) = y_p(x) + y_c(x)$$

$$a\frac{d^2}{dx^2}[y_p + y_c] + b\frac{d}{dx}[y_p + y_c] + c[y_p + y_c] = f(x)$$

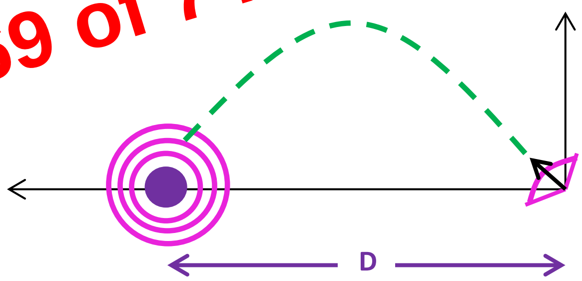
$$a\frac{d^2y_c}{dx^2} + b\frac{dy_c}{dx} + cy_c + a\frac{d^2y_p}{dx^2} + b\frac{dy_p}{dx} + cy_p = f(x)$$

$$0 + a\frac{d^2y_p}{dx^2} + b\frac{dy_p}{dx} + cy_p = f(x)$$

Solving differential equations

E.g.

Arrow mass = m



Equation of motion: $m\frac{d^2\bar{R}}{dt^2} = m\bar{g}$

In components: $\frac{d^2x}{dt^2} = 0$ $\frac{d^2y}{dt^2} = -g$

Solve the y components

1) Find the general solution of $\frac{d^2y}{dt^2} = 0$

Auxiliary equation: $\alpha^2 = 0$ → $\alpha = 0$

$$y_c(t) = \lambda_1 + \lambda_2 t$$

2) Particular solution:
Try: $y_p(t) = D_0 + D_1 t + D_2 t^2$

$$\frac{d^2y_p}{dx^2} = 2D_2 = -g$$

$$D_2 = -\frac{1}{2}gt^2$$