then S is an interval

Nested Intervals Property (2.5.2)

If $I_n = [a_n, b_n]$, n ∈ N, is a nested sequence of closed bounded intervals, then there exists a number ξ ∈ R such that $\xi \in I_n$ for all n ∈ N.

Theorem 2.5.3

If $I_n = [a_n, b_n]$, n ∈ N, is a nested sequence of closed bounded intervals such that the lengths $b_n - a_n$ of I_n satisfy inf{$b_n - a_n$: n ∈ N} = 0, then the number ξ contained in I_n for all n ∈ N is unique

Theorem 2.5.4

The set R of real numbers is not countable

Theorem 2.5.5

The unit interval [0, 1] := {x in R: 0 ≤ x ≤ 1} is not countable

Sequence of Real Numbers (Definition 3.1.1)

A sequence of real numbers is a function defined on the set N = {1, 2, … } of natural numbers whose range is contained in the set R of real numbers

Definition 3.1.3

A sequence X = (x_n) ∈ R is said to **converge** to x ∈ R, or x is said to be a **limit** of (x_n), if for every ε > 0 there exists a natural number K(ε) such that for all n ≥ K(ε), the terms x_n satisfy $|x_n - x| < \varepsilon$

If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit, we say that the sequence is **divergent**

Uniqueness of Limits (3.1.4)

A sequence in R can have at most one limit

Theorem 3.1.5

Let X = (x_n) be a sequence of real numbers, and let x in R. The following statements are equivalent:

(a) X converges to x

(b) For every ε > 0, there exists a natural number K such that for all n ≥ K, the terms x_n satisfy $|x_n - x| < \varepsilon$

(c) For every ε > 0, there exists a natural number K such that for all n ≥ K, the terms x_n satisfy $x - \varepsilon < x_n < x + \varepsilon$

(d) For every ε- neighborhood V_ε(x) of x, there exists a natural number K such that for all n ≥ K, the terms x_n belong to V_ε(x)

m-tail (Definition 3.1.8)

If X = (x_1, x_2, …, x_n, ...) is a sequence of real numbers and if m is a given natural number, then the m-tail of X is the sequence X_m := (x_{m+n}: n ∈ N) = (x_{m+1}, x_{m+2}, …)

Theorem 3.1.9

Let X = (x_n: n ∈ N) be a sequence of real numbers and let m ∈ N. Then the m-tail X_m= (x_{m+n}: n ∈ N) of X converges if and only if X converges. In this case, lim X_m = lim X

(a) The **limit superior** of (x_n) is the infimum of the set V of v in R such that v < x_n for at most a finite number of n in N
(b) The **limit inferior** of (x_n) is the supremum of the set of w in R such that x_m < w for at most a finite number of m in N.
An alternative definition of lim sup(xn) (3.4.11 (c)):

If u_m = sup{x_n: n ≥ m}, then x* = inf{u_m: m in N} = lim(u_m)

Theorem 3.4.11

If (x_n) is a bounded sequence of real numbers, then the following statements for a real number x* are equivalent:

(a) x* = lim sup(x_n)
(b) If ε > 0, there are at most a finite number of n in N such that x* + ε < x_n, but an infinite number of n in N such that x* - ε < x_n
(c) If u_m = sup{x_n: n ≥ m}, then x* = inf{u_m: m in N} = lim(u_m)
(d) If S is the set of subsequential limits of (x_n), then x* = sup S

Theorem 3.4.12

A bounded sequence (x_n) is convergent if and only if lim sup (x_n) = lim inf (x_n)

Cauchy Sequence (Definition 3.5.1)

A sequence X = (x_n) of real numbers is said to be a **Cauchy Sequence** if for every ε > 0 there exists a natural number H(ε) such that for all natural numbers n, m ≥ H(ε), the terms x_n, x_m satisfy |x_n - x_m| < ε

Lemma 3.5.3

If X = (x_n) is a convergent sequence of real numbers, then X is a Cauchy sequence

Lemma 3.5.4

A Cauchy sequence of real numbers is bounded

Cauchy Convergence Criterion (3.5.5)

A sequence of real numbers is convergent if and only if it is a Cauchy sequence

Contractive (definition 3.5.7)

We say that a sequence X = (x_n) of real numbers is contractive if there exists a constant C, 0 < C < 1, such that |x_{n+2} - x_{n+1}| ≤ C|x_{n+1}- x_n| for all n in N. The number C is called the constant of the contractive sequence.

Theorem 3.5.8

Every contractive sequence is a Cauchy sequence, and therefore is convergent

Definition 3.6.1

Let (x_n) be a sequence of real numbers.
(i) We say that (x_n) **tends to** $\pm$∞, and write lim(x_n) = +∞, if for every α ∈ R there exists a natural number K(α) such that if n ≥ K(α), then x_n > α
(ii) We say that (x_n) **tend to** -∞, and write lime(x_n) = -∞, if for every β ∈ R there exists a natural number K(β) such that n ≥ K(β), then x_n < β
We say that (x_n) is **properly divergent** in case we have either

lim(x_n) = +∞ or lim (x_n) = -∞