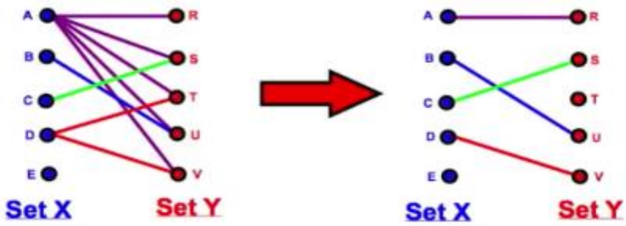
Matchings

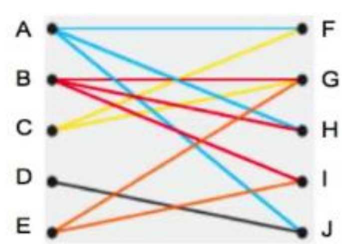
A **matching** is the one-to-one pairing of some or all of the elements of one set, **X**, with the elements of a second set, **Y**.

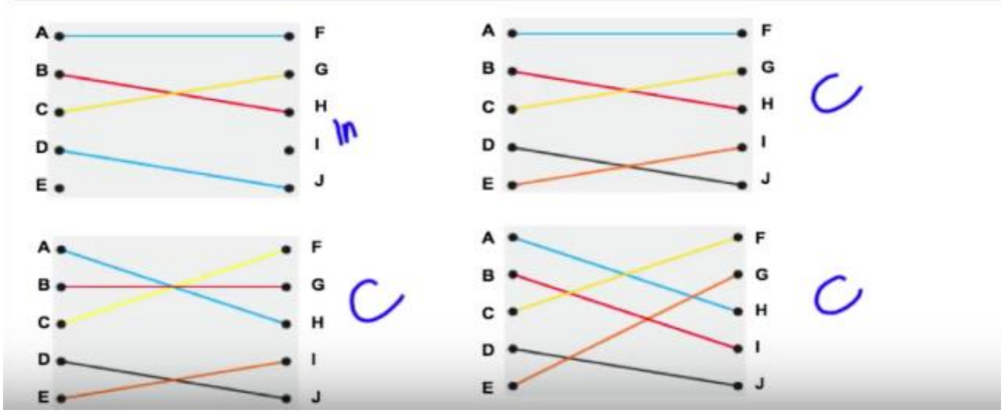


A **complete matching** is when every member of **X** is paired with one member of **Y**.

Your turn...

Identify 4 possible matchings from this bipartite graph.



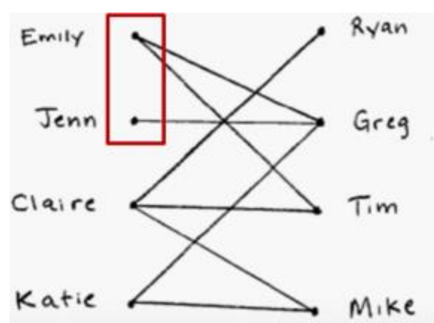


Hall's Marriage Theorem

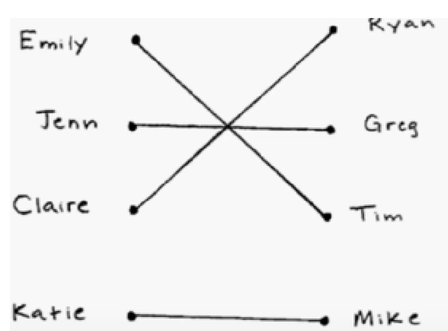
For a bipartite graph $G = (V, E)$, with bipartition (V_1, V_2), there exists a matching $M \subseteq E$ that covers V_1 if and only if for all $S \subseteq V_1$, $|S| \leq |N(S)|$.

Example - Dancing Problem

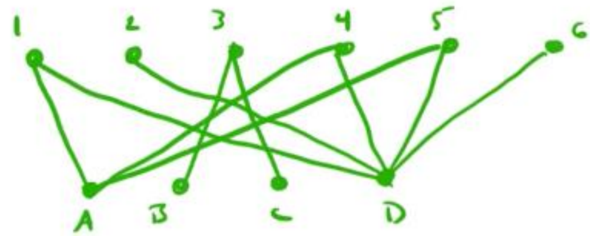
- Suppose that we are at a dance were there are 4 boys and 4 girls. Given the following information we want to use Hall's Theorem to prove that every person at the dance will be able to find a dance partner.



- The four boys are Ryan, Greg, Tim, and Mike
- Each boy is willing to dance with any girl that is willing to dance with him
- The four girls are Emily, Jenn, Claire, and Katie.
  - Emily is willing to dance with Greg and Tim
  - Jenn is willing to dance with Greg
  - Claire is willing to dance with Ryan, Mike and Tim
  - Katie is willing to dance with Mike and Greg.



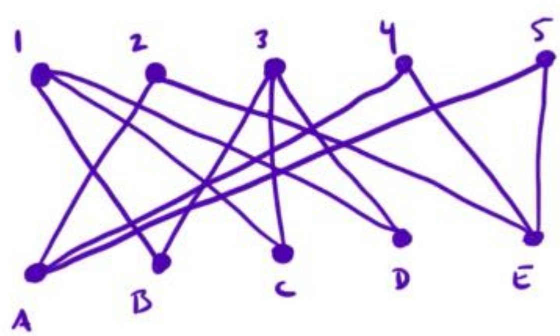
ex: Does G have a matching of size 4?



let $X = \{B, C\}$. $N(X) = \{3\}$. $|N(X)| = 1$ $|X| = 2$

violates Hall's condition.

ex: Does G have a matching of size 5?



let $X = \{B, C, D\}$. $N(X) = \{1, 3\}$. $|N(X)| \not\geq |X|$ violates Hall's condition NO.

$2 \not\geq 3$

Isomorphism

- The word *isomorphism comes from the Greek roots isos for "equal" and morphe for "form."*
- The simple graphs *G1 = (V1,E1) and G2 = (V2,E2) are* ***isomorphic*** *if there exists a bijection (***one-to-one*** and **onto)** function *f from V1 to V2*
  - *with the property that a and b are adjacent in G1 if* and only if *f (a) and f (b) are adjacent in G2, for all a and b in V1.*

Isomorphism of Graphs

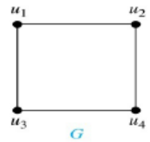
Definition: Two (undirected) graphs *G1* = (*V1*, *E*1) and *G2* = (*V2*, *E2*) are *isomorphic* if there is a bijection, $f: V_1 \to V_2$, with the property that for all vertices $a, b \in V_1$

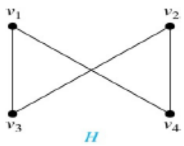
$$\{a,b\} \in E_1 \quad \text{if and only if} \quad \{f(a), f(b)\} \in E_2$$

Such a function *f* is called an *isomorphism*.
Intuitively, isomorphic graphs are "THE SAME", except for "renamed" vertices.

Isomorphism of Graphs (*cont.*)

Example: Show that the graphs G =(V, E) and H = (W, F) are isomorphic.





Solution: The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one correspondence between V and W. To see that this correspondence preserves adjacency, note that adjacent vertices in G are u_1 and u_2, u_1 and u_3, u_2 and u_4, and u_3 and u_4, and each of the pairs $f(u_1) = v_1$ and $f(u_2) = v_4$, $f(u_1) = v_1$ and $f(u_3) = v_3$, $f(u_2) = v_4$ and $f(u_4) = v_2$, and $f(u_3) = v_3$ and $f(u_4) = v_2$ consists of two adjacent vertices in H. ◀

Isomorphism of Graphs (*cont.*)

It is difficult to determine whether two graphs are isomorphic by brute force: there are n! bijections between vertices of two n-vertex graphs.

Often, we can show two graphs are not isomorphic by finding a property that only one of the two graphs has. Such a property is called *graph invariant*:

- e.g., number of vertices of given degree, the degree sequence (list of the degrees), ...

Paths in directed graphs (same definitions)

Definition: For an directed graph $G = (V, E)$, an integer $n \geq 0$, and vertices $u, v \in V$, a path (or walk) of length n from u to v in G is a sequence of vertices and edges $x_0, e_1, x_1, e_2, \ldots, x_n, e_n$, such that $x_0 = u$ and $x_n = v$, and such that $e_i = (x_{i-1}, x_i) \in E$ for all $i \in \{1, \ldots, n\}$.

When there are no multi-edges in the directed graph G, the path can be denoted (uniquely) by its vertex sequence $x_0, x_1, \ldots, x_n$.

A path of length $n \geq 1$ is called a circuit (or cycle) if the path starts and ends at the same vertex, i.e., $u = v$.

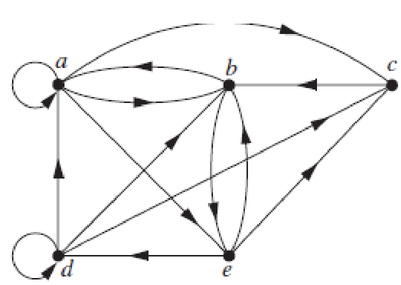
A path or circuit is called simple if it does not contain the same edge more than once. (And we call it tidy if it does not contain the same vertex more than once, except possibly the first and last in case $u = v$ and the path is a circuit (cycle).)

Path-example

Which of the following are paths in the directed graph shown in Figure 1:

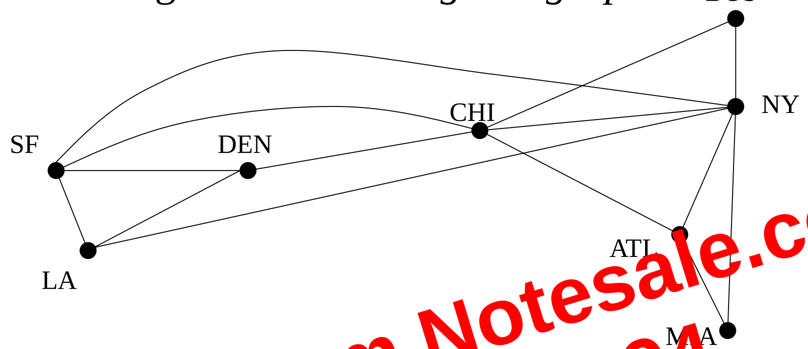
a, b, e, d;
a, e, c, d, b;
b, a, c, b, a, a, b;
d, c; *c, b, a*;
e, b, a, b, a, b, e?

What are the lengths of those that are paths? Which of the paths in this list are circuits?



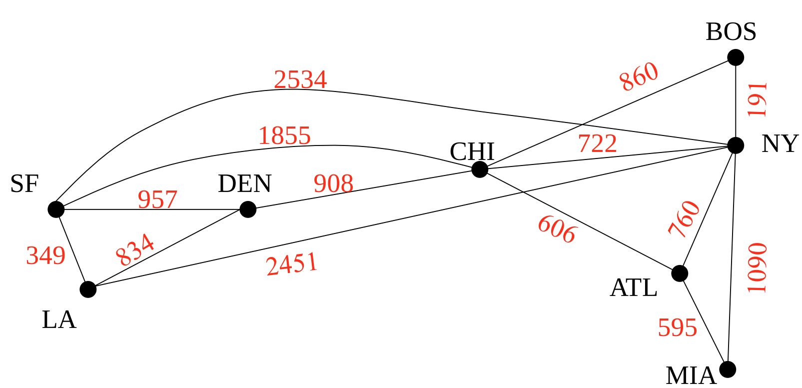
Weighted Graphs

Graphs that have a number assigned to each edge are called *weighted graphs*.



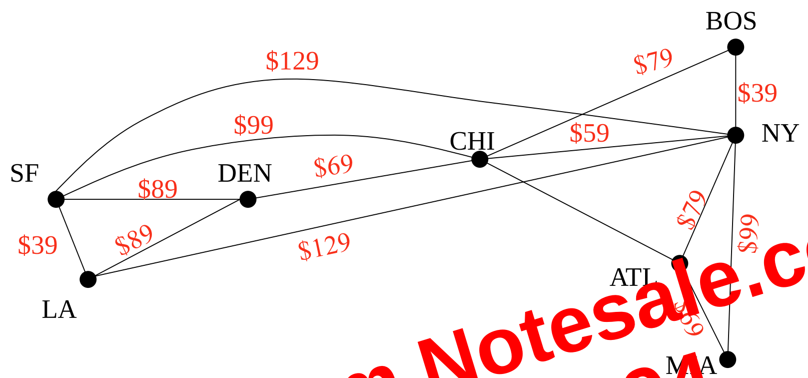
Weighted Graphs

MILES



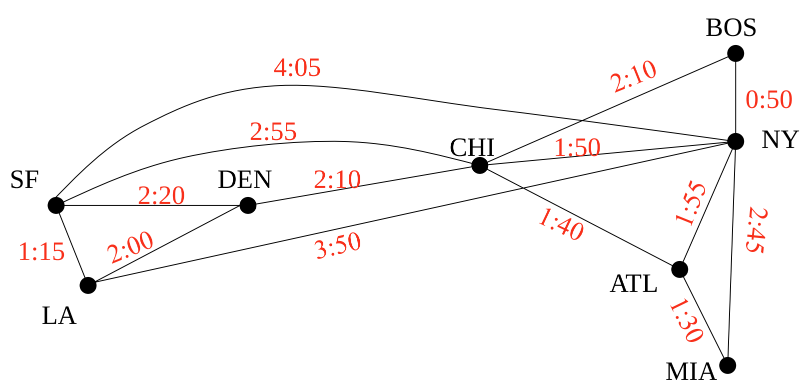
Weighted Graphs

FARES



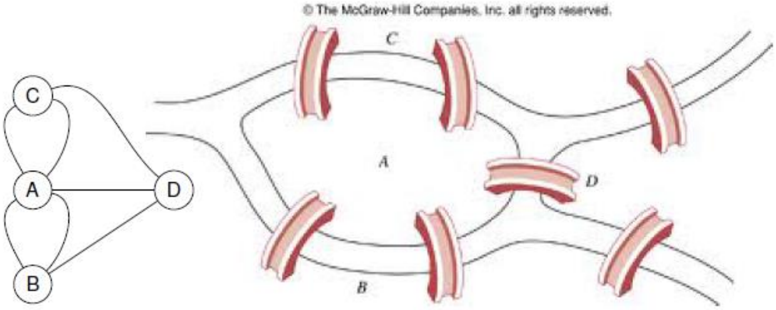
Weighted Graphs

FLIGHT
TIMES



The Königsberg Bridge Problem

Leonard Euler (1707-1783) was asked to solve the following:



The question is whether it is possible to walk with a route that crosses each bridge (edge) exactly once, and return to the starting point.

Question: Can you start a walk somewhere in Königsberg, walk across each of the 7 bridges exactly once, and end up back where you started from?

Euler (in 1736) used "graph theory" to answer this question.

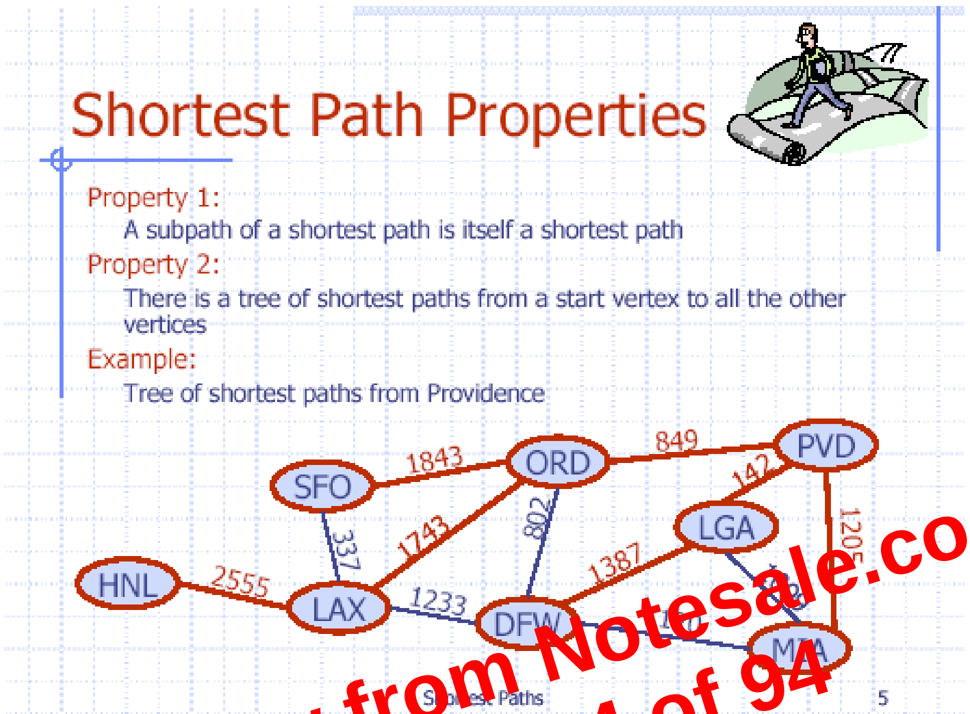
Euler paths and Euler Circuits

Recall that an (undirected) multigraph does not have any loops, but can have multiple edges between the same pair of vertices.

Definition: An Euler path in a multigraph *G* is a simple path that contains every edge of *G*.
(So, every edge occurs exactly once in the path.)

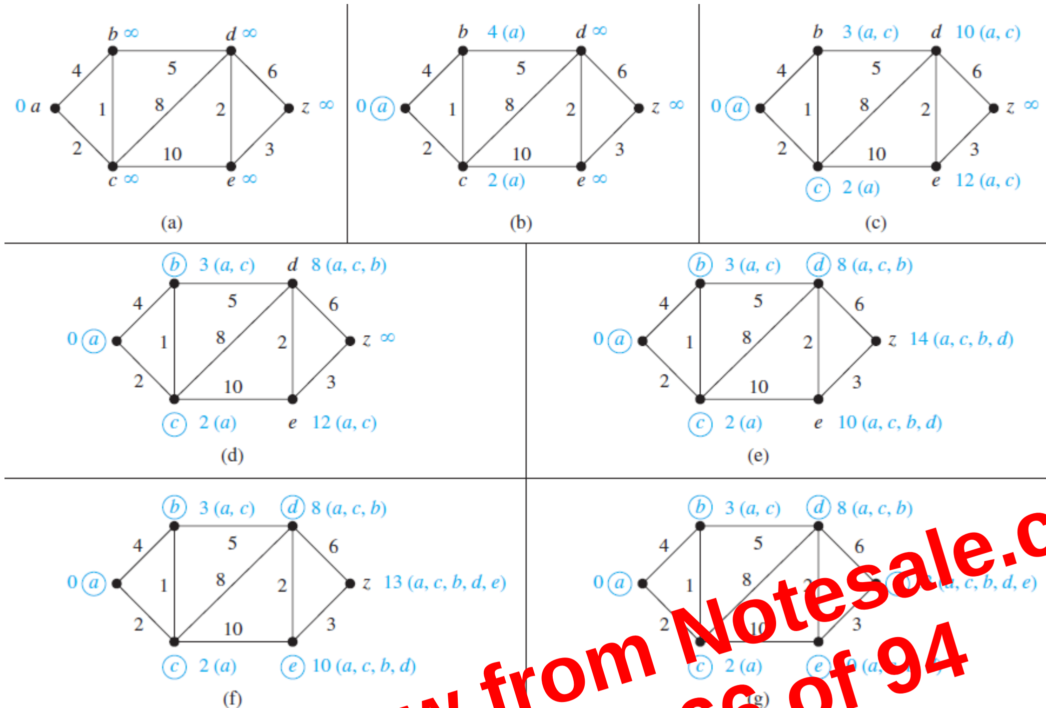
An Euler circuit in an multigraph *G* is a simple circuit that contains every edge of *G*.
(So, every edge occurs exactly once in the circuit.)

Question: Is there a simple criterion for determining whether a multigraph *G* has an Euler path (an Euler circuit)?
Answer: Yes. Euler's Theorem



Dijkstra's Algorithm

- Dijkstra's algorithm is used in problems relating to finding the shortest path.
- Each node is given a temporary label denoting the length of the shortest path *from* the start node *so far*.
- This label is replaced if another shorter route is found.
- Once it is certain that no other shorter paths can be found, the temporary label becomes a permanent label.
- Eventually all the nodes have permanent labels.
- At this point the shortest path is found by retracing the path backwards.



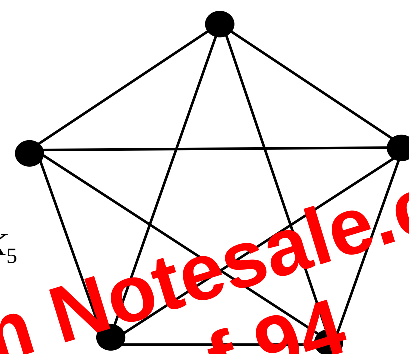
Dijkstra's Algorithm

- Dijkstra's algorithm is used in problems relating to finding the shortest path.
- Each node is given a temporary label denoting the length of the shortest path *from* the start node *so far*.
- This label is replaced if another shorter route is found.
- Once it is certain that no other shorter paths can be found, the temporary label becomes a permanent label.
- Eventually all the nodes have permanent labels.
- At this point the shortest path is found by retracing the path backwards.

Euler’s Formula (Cont.)

- Corollary 1: If G is a connected planar simple graph with e edges and v vertices where $v \geq 3$, then $e \leq 3v - 6$.
- Is K_5 planar?

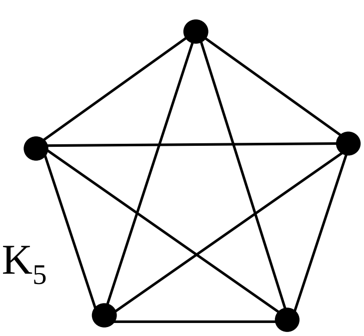
K_5



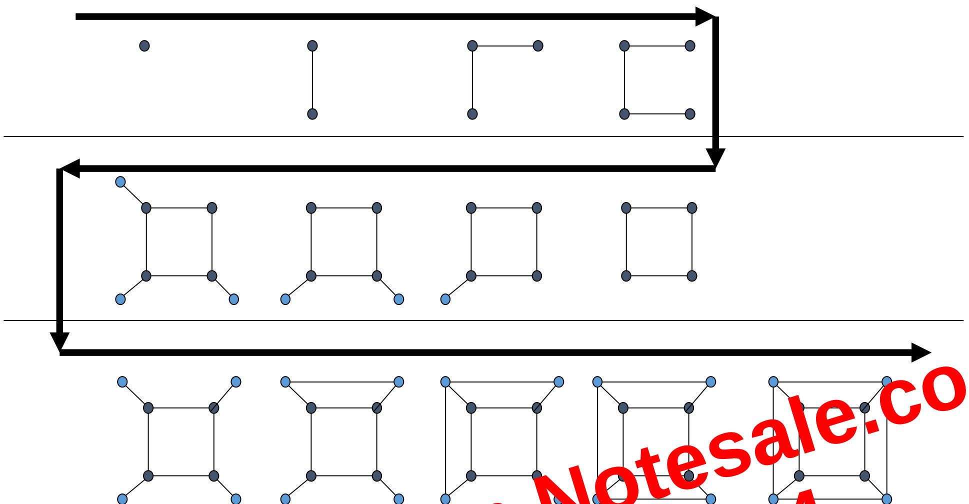
Euler’s Formula (Cont.)

- K_5 has 5 vertices and 10 edges.
- We see that $v \geq 3$.
- So, if K_5 is planar, it must be true that $e \leq 3v – 6$.
- $3v – 6 = 3*5 – 6 = 15 – 6 = 9$.
- So e must be ≤ 9.
- But $e = 10$.
- So, K_5 is nonplanar.

K_5



Euler Characteristic



Euler Characteristic

Thus to prove that χ is always 2 for planar graphs, one calculate χ for the trivial vertex graph:

$$\chi = 1\text{-}0\text{+}1 = 2$$

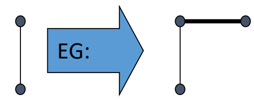
and then checks that each possible move does not change χ .

Euler Characteristic

Check that moves don't change χ :

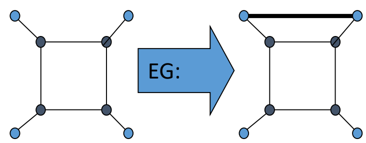
1) Adding a degree 1 vertex:

r is unchanged. $|E|$ increases by 1. $|V|$ increases by 1. χ += (0-1+1)



2) Adding an edge between pre-existing vertices:

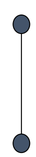
r increases by 1. $|E|$ increases by 1. $|V|$ unchanged. χ += (1-1+0)



Animated Invariance of Euler Characteristic

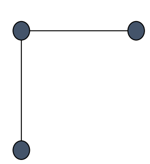
$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
1	0	1	2

Animated Invariance of Euler Characteristic



$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
2	1	1	2

Animated Invariance of Euler Characteristic



$\|V\|$	$\|E\|$	r	$\chi = r - \|E\| + \|V\|$
3	2	1	2