Unit 2

Techniques of Integration

Example 2.0

Evaluate the integral.

1. $\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C$

2. $\int \sin(3x)\,dx = -\frac{1}{3}\cos(3x) + C$

3. $\int 2\sec^2(5x)\,dx = \frac{2}{5}\tan(5x) + C$

4. $\int \csc(7x)\cot(7x)\,dx = -\frac{1}{7}\csc(7x) + C$

End of Unit 2.0

$$\int x^2 e^x \, dx = x^2 e^x - 2\int xe^x \, dx$$

$$\int xe^x \, dx = xe^x - e^x + C$$

Therefore,

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C_1$$

4. $\int e^x \cos x \, dx$

$= \int u \, dv$

$= uv - \int v \, du$

$= e^x \sin x - \int e^x \sin x \, dx$

Let	$u = e^x$	$dv = \cos x \, dx$
	$du = e^x dx$	$v = \sin x$

Required Exercises

Answer Exercises in TC7.

Exercises 7.1 (1-32) on page 582

Check your answers on A-159.

$$= -\int \left(1-u^2\right)^2 u^2\, du$$

$$= -\int \left(1-2u^2+u^4\right)u^2\, du$$

$$= -\int \left(u^2-2u^4+u^6\right)du$$

$$= -\left(\frac{u^3}{3}-\frac{2u^5}{5}+\frac{u^7}{7}\right)+C$$

$$= -\frac{\cos^3 x}{3}+\frac{2\cos^5 x}{5}-\frac{\cos^7 x}{7}+C$$

Example 2.2.3

Evaluate the integrals.

1. $\int \tan^2(2x)\sec^4(2x)\,dx$

$$= \int \tan^2(2x)\sec^2(2x)\sec^2(2x)\,dx$$

$$= \int \tan^2(2x)\left(1+\tan^2(2x)\right)\sec^2(2x)\,dx$$

$$= \frac{1}{2}\int u^2\left(1+u^2\right)du$$

$$= \frac{1}{2}\int \left(u^2+u^4\right)du$$

$$= \frac{1}{2}\left(\frac{u^3}{3}+\frac{u^5}{5}\right)+C$$

Let $u = \tan(2x)$

$$du = 2\sec^2(2x)\,dx$$

$$\frac{du}{2} = \sec^2(2x)\,dx$$

Preliminary Exercise

Express the following in terms of $\sin\theta$ and $\cos\theta$.

1. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

2. $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

3. $\sec\theta = \dfrac{1}{\cos\theta}$

4. $\csc\theta = \dfrac{1}{\sin\theta}$

Express the following in terms of $\tan\theta$ and $\sec\theta$.

1. $\cot\theta = \dfrac{1}{\tan\theta}$

2. $\cos\theta = \dfrac{1}{\sec\theta}$

3. $\sin\theta = \dfrac{\tan\theta}{\sec\theta}$

4. $\csc\theta = \dfrac{\sec\theta}{\tan\theta}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$= \frac{1}{25}\int \cos\theta \, d\theta$$

$$= \frac{1}{25}\sin\theta + C$$

$$= \frac{1}{25}\cdot\frac{\tan\theta}{\sec\theta} + C$$

$$= \frac{1}{25}\cdot\frac{x}{\sqrt{x^2+25}} + C$$

$$x = 5\tan\theta$$

$$\sqrt{x^2+25} = 5\sec\theta$$

When to Use Partial Fractions

Consider $\int \frac{P}{Q}\,dx.$

Conditions:

1. P and Q are polynomials.
2. $\deg P < \deg Q$. Otherwise, divide P by Q.
3. P and Q have no common factors.
   Otherwise, cancel common factors.

Case 1: Distinct Linear Factors

If Q has distinct linear factors,

$$Q = (a_1 x + b_1)(a_2 x + b_2)\cdots(a_n x + b_n)$$

write $$\frac{P}{Q} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_n}{a_n x + b_n}$$

$$\frac{4x+3}{2x^2+3x+1} = \frac{A}{x+1} + \frac{B}{2x+1}$$

$A = 1$ and $B = 2$

Therefore, $$\frac{4x+3}{2x^2+3x+1} = \frac{1}{x+1} + \frac{2}{2x+1}.$$

$$\int \frac{4x+3}{2x^2+3x+1} dx = \int \left(\frac{1}{x+1} + \frac{2}{2x+1} \right) dx$$

$$= \ln|x+1| + 2 \cdot \frac{1}{2} \ln|2x+1| + C$$

$$=-2\int\frac{\left(u^3+2u\right)du}{u-5}$$

$$\boxed{u=\sqrt{x^2-2}}$$

$$=-2\int\left(u^2+5u+27+\frac{135}{u-5}\right)du$$

$$=-2\left(\frac{u^3}{3}+\frac{5u^2}{2}+27u+135\ln\left|u-5\right|\right)+C$$

$$=\frac{-2}{3}\left(x^2-2\right)^{3/2}-5\left(x^2-2\right)-54\sqrt{x^2-2}$$

$$-270\ln\left|\sqrt{x^2-2}-5\right|+C$$

End of Unit 2.5

2. $$\int \frac{dx}{\sin x + \tan x}$$

$$= \int \frac{dx}{\sin x + \dfrac{\sin x}{\cos x}}$$

$$= \int \frac{\dfrac{2\,dz}{1+z^2}}{\dfrac{2z}{1+z^2} + \dfrac{\dfrac{2z}{1+z^2}}{\dfrac{1-z^2}{1+z^2}}}$$

Let $z = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2dz}{1+z^2}$$

$$\sin x = \frac{2z}{1+z^2}$$

$$\cos x = \frac{1-z^2}{1+z^2}$$