The force per unit length is thus given by

$$F = \int dA\,\Delta p = \int_{-b/2}^{b/2} dx\,\rho U_0^2 \frac{a\pi}{b}\cos\left(\frac{\pi x}{b}\right) = 2a\rho U_0^2$$

By the Kutta-Joukowski Theorem:

$$F = -\rho U_0 \Gamma \quad \longrightarrow \quad \Gamma = -2aU_0$$

This means that the circulation is negative (resulting in lift), and that there must be a trailing vector on the edge of the wing.

$$F = 2(0.1)(2)^2(10^3) = 800\,\mathrm{Nm}^{-1}$$

Evaluating the Reynolds number for this flow:

$$\mathrm{Re} = \frac{U_0\ell_0}{\nu} \backsim \frac{U_0 a}{\nu} \backsim 10^5 \gg 10^3$$

This means that the Laminar flow assumption is not valid, as the flow is turbulent.

Question 2

Consider the Navier Stokes equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + \eta\nabla^2\mathbf{u} - \nabla\chi$$

Assume that the flow is steady ($\partial_t \mathbf{u} = 0$), viscous ($\nu \gg 1$), and ignore the effects of gravity ($\chi = 0$). As $\nu \gg 1$, Re $\ll 1$, meaning that we can neglect the inertial terms in the Navier Stokes. Then:

$$\frac{1}{\rho}\nabla p = \nu\nabla^2\mathbf{u}$$

Assume that the flow is only along x (we can neglect flow along y and z due to symmetry constraints). Then, as $\nabla p = -f$, we have that

$$\frac{1}{\rho}(-f) = \nu\nabla^2 u = \nu\frac{\partial u}{\partial z^2}$$

assuming that the flow varies along z (cannot vary along y due to symmetry, nor along x due to the fact that the flow is steady, and well-developed). Integrating:

$$u = \frac{1}{2\rho\nu}\frac{\partial p}{\partial x}z^2 + c_1 z + c_2$$

Boundary conditions: $u = 0$ at $z = 0$ and $z = b$.

$$\begin{aligned} c_2 &= 0 \\ c_1 &= -\frac{1}{2\rho\nu}\frac{\partial p}{\partial z}b \\ \rightarrow u &= \frac{1}{2\rho\nu}\frac{\partial p}{\partial z}z(z-b) = -\frac{f}{2\rho\nu}z(z-b) \end{aligned}$$

This is a valid solution for the flow if Re $\gg 1$, meaning that we require that

$$\nu \ll u_0\ell_0 = b\frac{fb^2}{8\rho\nu} \quad \longrightarrow \quad \nu^2 \ll \frac{fb^3}{8\rho}$$

The remainder of this question appears to be no longer on syllabus, so I will not provide any solution here such that

$$
\begin{aligned}
-\lambda(a-\lambda)+1&=0\\
-\lambda a+\lambda^2+1&=0\\
\lambda&=\frac{a}{2}\pm i\sqrt{1-\left(\frac{a}{2}\right)^2}
\end{aligned}
$$

As $a \ll 1$, the quantity inside the square root is positive, so the fixed point is clearly an unstable spiral. Confinement in the x-y plane will break down when $x \backsim c$, as this means that we can no longer neglect the former term.

Consider

$$
\Lambda=\frac{1}{2}(\mathcal{J}+\mathcal{J}^T)=\begin{pmatrix} 0 & 0 & \frac{1}{2}(z-1)\\ 0 & a & 0\\ \frac{1}{2}(z-1) & 0 & x-c\end{pmatrix}
$$

$\lambda_2=a$, meaning that displacements in the y direction grow exponentially. Considering the symmetrised Jacobean in the x-z plane:

$$
\Lambda_{xz}=\begin{pmatrix} 0 & \frac{1}{2}(z-1)\\ \frac{1}{2}(z-1) & x-c\end{pmatrix}\rightarrow \lambda_1,\lambda_3=\frac{1}{2}(x-c)\pm\frac{1}{2}\sqrt{(x-c)^2+(z-1)^2}
$$

Thus, as $\lambda_2>0$, either λ_1 or λ_3 is less than zero. Thus, errors will grow in two directions, and shrink in another, unless $z=1$. Now for $x=c$,

$$
\Lambda_{xz}=\begin{pmatrix} 0 & \frac{1}{2}(z-1)\\ \frac{1}{2}(z-1) & 0\end{pmatrix}
$$

This allows us to conclude that

$$
\begin{aligned}
\lambda_1&=\frac{1}{2}(z-1), \quad \mathbf{e}_1=(1,0,1) \qquad \text{fastest growth}\\
\lambda_3&=-\frac{1}{2}(z-1), \quad \mathbf{e}_2=(1,0,-1) \qquad \text{fastest decay}\\
\lambda_2&=a, \quad \mathbf{e}_3=(0,1,1)
\end{aligned}
$$

Question 4

I am not going to provide solutions to this question, given that it is completely un-enlightening to solve, and will not really inform any revision on the subject. There are also some departmental solutions online if the reader very much desires to see it solved.

Assuming no forcing ($\mathbf{f} = 0$), it si clear that this equation does not depend explicit on time, and only on space:

$$\nabla p(\mathbf{r}, t) = \eta \nabla^2 \mathbf{u}(\mathbf{r}, t)$$

But

$$\nabla p(\mathbf{r}, -t) = \eta \nabla^2 \mathbf{u}(\mathbf{r}, -t)$$

will also describe the same dynamical behaviour, meaning that such flows are time reversible.

Now, consider the dimensions of the relevant quantities:

$$\begin{aligned} [D] = [F] &= MLT^{-2} \\ [\eta] &= ML^{-1}T^{-1} \\ [a] &= L \\ [u_0] &= LT^{-1} \end{aligned}$$

It is thus clear that

$$D = \kappa a \eta u_0$$

for some constant of proportionality κ. Thus, as given by the question,

$$D = 6\pi a \eta u_0$$

We can use NII to write the equation of motion of the drop:

$$m\frac{d\mathbf{u}}{dt} = m\mathbf{g} - 6\pi a\eta\mathbf{u}$$

At terminal velocity, $d\mathbf{u}/dt = 0$, such that

$$6\pi a \eta u_t = \frac{4\pi}{3}a^3(\rho - \rho_0)g \quad \longrightarrow \quad u_t = \frac{2a^2}{9m}(\rho - \rho_0)g$$

where $\rho \approx 10^3$ kgm^{-3} and $\rho_0 \approx 1.2$kgm^{-3} are the density of the water droplet and air respectively. We thus estimate that $u_t \approx 0.012\,\text{ms}^{-1}$. Evaluating the Reynolds number:

$$\text{Re} = \frac{u_0 \ell_0}{\nu} \backsim \frac{u_t a}{\rho_0 \eta} \backsim 0.005$$

meaning that we are indeed in the low Reynolds number limit.

Consider NII:

$$m\ddot{\mathbf{x}} = \sum \mathbf{F} = \text{Drag} + \text{Stochastic Forces} = -\gamma\dot{\mathbf{x}} + \mathbf{F}(t)$$

where $\gamma = 6\pi\eta a$, and $\mathbf{F}$ is some stochastic field that satisfies

$$\left\langle \mathbf{F}(t) \cdot \mathbf{F}(t') \right\rangle = 2\kappa\delta(t - t')$$

for some characteristic constant κ. Physically, this delta-correlated field is due to the random thermal fluctuations of the molecules surrounding our test mass m that 'bumps' it around.

For small Re flows, viscous effects are dominant, and so we ignore the inertial terms. Then, the Navier Stokes equation becomes

$$\frac{1}{\rho}(\nabla p + \nabla \chi) = \nu \nabla^2 \mathbf{u}$$

Assuming that χ is the conservative potential associated with the gravitational force, so $\chi = \rho g z$. We thus arrive at

$$\nabla(p + \rho g z) = \rho\nu\nabla^2 \mathbf{u}$$

as required. $\mathbf{u}$ will vary most along z as $\partial^2/\partial z^2 \backsim 1/h^2$, which is a small quantity; derivatives along z dominate. Furthermore, we have that

$$\frac{\partial}{\partial z}(\nabla_h p) = 0$$

due to the fact that the flow must be well-developed. Taking the horizontal component of our Stokes' equation, we have that

$$\frac{\partial^2 u_h}{\partial z^2} = \frac{1}{\rho\nu}\nabla_h p$$

which has solution

$$u_h = \frac{1}{\rho\nu}\nabla_h p z^2 + c_1 z + c_2$$

for constants c_1 and c_2. We have the boundary conditions

$$\begin{aligned} u_h &= 0 \quad \text{at} \quad z = 0 \quad \longrightarrow \quad c_2 = 0 \\ u_h &= 0 \quad \text{at} \quad z = h \end{aligned}$$

such that

$$u_h = -\frac{1}{2\rho\nu} z(h - z)\nabla_h p$$

as required. Now, by incompressibility, we have that

$$\nabla \cdot \mathbf{u} = \nabla_h \cdot u_h + \frac{\partial u_z}{\partial z} = 0$$

However, as h is small, we argue that there cannot be a large flow along z, so $u_z \backsim 0$. This means that

$$\nabla_h \cdot u_h = 0 - \frac{1}{2\rho\nu} z(h - z)\nabla_h^2 p \quad \longrightarrow \quad \nabla_h^2 p = 0$$

The vertical component of the Stokes' equation reads

$$\frac{\partial p}{\partial z} = \rho g \quad \longrightarrow \quad p_z = -\rho g z + c_3$$

The solution to Laplace's equation in cylindrical polar coordinates is of the form

$$p_h = \left(c_1 r + \frac{c_2}{r}\right)\cos\theta$$

meaning that a full solution is of the form

$$p = -p_1\left(r + \frac{a^2}{r}\right)\cos\theta - \rho g z + p_0$$

which has solutions

$$\lambda = -b, \quad \lambda = -\frac{1}{2}\left[(1+\sigma) \pm \sqrt{1-(2-4r)\sigma+\sigma^2}\right]$$

Thus, trajectories along z converge to zero at an exponentially fast rate. In the case of the second set of eigenvalues, these are clearly all negative for $r < 1$, and thus stable. For $r > 1$, then this is clearly a saddle point, meaning that a supercritical pitchfork bifurcation clearly occurs at $r_c = 1$.

Consider a volume element

$$\delta V = \delta x \delta y \delta z$$

Then, clearly,

$$\frac{\delta \dot{V}}{\delta V} = \frac{\delta \dot{x}}{\delta x} + \frac{\delta \dot{y}}{\delta y} + \frac{\delta \dot{z}}{\delta z} \approx \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} = -(1+\sigma+b)$$

This means that volumes will evolve as

$$\delta V = \delta V_0 e^{-(1+\sigma+b)t}$$

That is, the volume elements in phase space converge to zero. This implies the existence of a strange attractor contained within a finite trapping region.

A positive Lyapanov exponent means that trajectories in phase space will separate exponentially quickly, but these will remain confined within some finite region of space (as volumes contract). This implies that the geometry of the phase space must be fractal in nature (stretching and folding, self similar at all scales). A positive Lyapanov exponent also implies chaotic behaviour, characterised by

- Aperiodic long term behaviour
- Sensitive dependence on initial conditions
- Deterministic, despite its unpredictable nature

Evaluating the required expressions, it trivially follows that

$$\begin{aligned}
\dot{e}_x &= \sigma(e_y - e_x) \\
\dot{e}_y &= -e_y - xe_z \\
\dot{e}_z &= xe_y - be_z \\
\dot{E} &= -2(e_x^2 + e_y^2 + be_z^2 - e_x e_y)
\end{aligned}$$

It is clear that $\dot{E} < 0$. This means that $e_x, e_y, e_z \to 0$ at long times, such that $x \to X$, $y \to Y$, $z \to Z$. If $x(t)$ is known, then the system of equations in X, Y and Z is no longer non-linear, and can be solved fully. Likewise, we can fully solve the e_x, e_y, e_z system, which will allow us to obtain the solutions to the Lorenz system at long times.

Now, we define the torque

$$T = -\frac{\partial U}{\partial \theta}$$

From the graphs, we see that

$$T_A = -\frac{\partial U_A}{\partial \theta} = \begin{cases} -U_0/\pi & \text{for} \quad s < \theta < s+\pi \\ U_0/\pi & \text{for} \quad s+\pi < \theta < s+2\pi \end{cases}$$

Then, the average torque is given by

$$\bar{T}(s) = \int_s^{s+2\pi} d\theta \, P_A(\theta) T_A(\theta) = \frac{U_0}{\pi}\left(-\int_s^{s+\pi} d\theta \, P_A(\theta) + \int_{s+\pi}^{s+2\pi} d\theta \, P_A(\theta)\right)$$

meaning that $T_0 = U_0/\pi$. Given the periodicity of our solutions, we can actually write this expression as

$$\bar{T}(s) = T_0 \int_s^{s+\pi} d\theta \, (P_A(\theta+\pi) - P_A(\theta))$$

To maximise the torque, use the fundamental theorem of calculus:

$$\begin{aligned} \bar{T}'(s_{\max}) &= T_0 \left[P_A(s_{\max}+2\pi) - P_A(s_{\max}+\pi) - P_A(s_{\max}+\pi) + P_A(s_{\max})\right] \\ &= 2T_0 \left[P_A(s_{\max}) - P_A(s_{\max}+\pi)\right] = 0 \end{aligned}$$

where we have again used the fact that the solution is periodic. This means that

$$P_A(s_{\max}) = P_A(s_{\max}+\pi)$$

But by definition

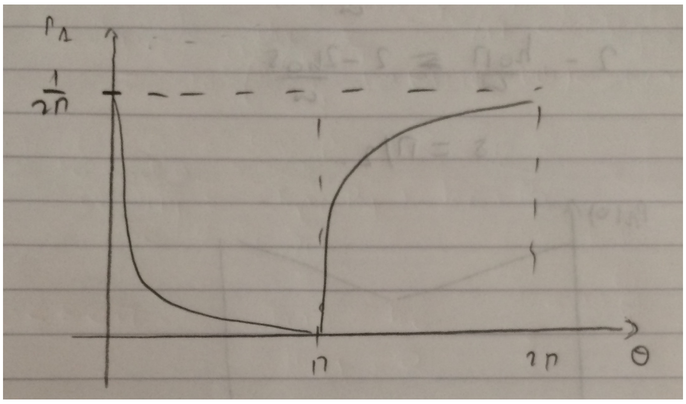
$$P_A(\theta) + P_A(\theta+\pi) = \frac{1}{2\pi}$$

such that

$$P_A(s_{\max}) = P_A(s_{\max}+\pi) = \frac{1}{4\pi}$$

Let us consider the two cases given:

- $\omega \ll k_0$: The solution is dominated by the exponential, such that



2014

Question 1

Irrotational: $\nabla \times \mathbf{u} = 0 \quad \longrightarrow \quad \mathbf{u} = \nabla\phi$
Incompressible: $\nabla \cdot \mathbf{u} = 0 \quad \longrightarrow \quad \nabla^2\phi = 0$
Thus, the velocity potential obeys Laplace's equation.

Consider the Navier Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla(p+\chi) = \nu\nabla^2\mathbf{u}$$

Inviscid, so $\nu = 0$, such that

$$\frac{\partial}{\partial t}(\nabla\phi) + (\nabla\phi\cdot\nabla)\nabla\phi + \frac{1}{\rho}\nabla(p+\chi) = 0$$

We now use the vector identity, such that

$$(\nabla\phi\cdot\nabla)\nabla\phi = (\nabla\times\nabla\phi)\times\nabla\phi + \frac{1}{2}\nabla(\nabla\phi\cdot\nabla\phi) = \frac{1}{2}\nabla(\nabla\phi\cdot\nabla\phi)$$

where we have used the fact that $\nabla\times\nabla\phi = 0$, such that

$$\nabla\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi\cdot\nabla\phi) + \frac{1}{\rho}(p+\chi)\right) = 0$$

Integrating, and letting $\chi = \rho g z$, we have that

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi\cdot\nabla\phi) + \frac{p}{\rho} + gz = \text{constant}$$

We have the following boundary conditions:

- At $z = H$:

$$\frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial z} = 0$$

  ie. that the motion of the upper surface is governed by the vertical component of the flow. We have assumed that the velocity changes are small, such that $d\eta/dt \approx \partial\eta/\partial t$.

- Assuming small velocities, and that $p = 0$ at the surface, such that

$$\frac{\partial\phi}{\partial t} + gz = \text{constant}$$

  along the surface. We can make a gauge choice to set the constant to zero, meaning that our second boundary condition at $z = H$ becomes

$$\frac{\partial\phi}{\partial t} + g\eta = 0$$

- At $z = 0$, we have that

$$\frac{\partial\phi}{\partial z} = 0$$

  as we require that there is no perpendicular of the flow at the lower boundary.

no longer neglect the term $\nu\nabla^2\mathbf{u}$ in the Navier Stokes.

$$\begin{aligned}\frac{\nabla p}{\rho} &\backsim \nu\nabla^2\mathbf{u}\\ \frac{p}{\rho\ell} &\backsim \frac{\nu u}{\ell^2}\\ \ell &\backsim \frac{\nu u\rho}{p}\end{aligned}$$

But $p/\rho \backsim c^2$, and the typical velocity scale of u is $u \backsim c$. We thus conclude that

$$\ell \backsim \frac{\nu}{c}$$

Letting $c \approx 330\,\mathrm{ms}^{-1}$, $\nu = 1.5\times10^{-5}\,\mathrm{m}^2\mathrm{s}^{-1}$, such that $\ell \approx 5\times10^{-8}\,\mathrm{m}$.

Question 3

Fixed points for $\dot{\mathbf{r}} = 0$, so

$$\begin{aligned}yz &= 0\\ x &= y\\ 1 - xy &= 1\\ x &= y = \pm1\\ z &= 0\end{aligned}$$

Thus, the fixed points occur at

$$(1,1,0),\quad (-1,-1,0)$$

Stability is given by the eigenvalues of $\mathcal{J}$, where

$$\mathcal{J} = \begin{pmatrix} 0 & z & y\\ 1 & -1 & 0\\ -y & -x & 0\end{pmatrix}$$

Finding eigenvalues:

$$\begin{aligned}\begin{vmatrix} -\lambda & z & y\\ 1 & -1-\lambda & 0\\ -y & -x & -\lambda\end{vmatrix} &\overset{!}{=} 0\\ \lambda^3 + \lambda^2 + \underbrace{(y^2 - z)}_{B}\lambda + \underbrace{y(y+x)}_{C} &= 0\end{aligned}$$

At the fixed points, it is clear that $A = 1$, $B = 1$, $C = 2$, meaning that $C > AB$. Thus, by the information given in the question, the roots have positive real parts. This means that the points are all unstable, though the exact behaviour is determined from the sign and magnitude of the imaginary part (if it exists).

Consider a volume element

$$\delta V = \delta x\delta y\delta z$$

Then, clearly,

$$\frac{\delta\dot{V}}{\delta V} = \frac{\delta\dot{x}}{\delta x} + \frac{\delta\dot{y}}{\delta y} + \frac{\delta\dot{z}}{\delta z} \approx \frac{\partial\dot{x}}{\partial x} + \frac{\partial\dot{y}}{\partial y} + \frac{\partial\dot{z}}{\partial z} = \nabla\cdot\mathbf{u}$$