# B2: Unofficial Past Paper Solutions (2011-2016)

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## 2011

### Question 1

Proper time - The time as defined by a clock following the coordinate of a particular object. In terms of the metric:

$$\tau = \frac{1}{c}\int \sqrt{-g_{\mu\nu}dx^{\mu}dx^{\nu}}$$

Rapidity - Defined as

$$\beta = \tanh\rho, \quad \gamma = \cosh\rho$$



Proper acceleration - The rate of change of the velocity of an object as measured in its instantaneous rest frame.

Consider the four acceleration:

$$\dot{\mathsf{U}} = \frac{d\mathsf{U}}{d\tau} = \frac{d}{d\tau}(\gamma(c, \mathbf{u}))$$

Now,

$$\frac{d\gamma}{d\tau} = \frac{dt}{d\tau}\frac{d\gamma}{dt} = \gamma\frac{d\gamma}{dt} = \gamma\frac{d\gamma}{d\mathbf{u}}\cdot\frac{d\mathbf{u}}{dt} = \gamma\mathbf{a}\cdot\frac{d\gamma}{d\mathbf{u}}$$

Now,

$$\frac{d\gamma}{d\mathbf{u}} = \frac{\gamma^3\mathbf{u}}{c^2}$$
$$\frac{d\gamma}{dt} = \gamma^3\frac{\mathbf{u}\cdot\mathbf{a}}{c^3}$$

Thus,

$$\dot{\mathsf{U}} = \gamma^2\left(\frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\gamma^2, \frac{\mathbf{u}\cdot\mathbf{a}}{c^2}\gamma^2\mathbf{u} + \mathbf{a}\right)$$

Thus,

$$\beta = \frac{\alpha t}{\sqrt{1-\beta^2}} \quad \longrightarrow \quad \frac{v}{c} = \frac{\alpha t}{\sqrt{1+(\alpha t)^2}}$$

In the second case, we are considering a photon powered rocket. In the rest frame of the launchpad, the photons have 4-vectors:

$$d\mathsf{P} = \alpha M_0 c dt \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

In the rest frame of the rocket,

$$d\mathsf{P}' = \Lambda d\mathsf{P} = \alpha M_0 c dt \begin{pmatrix} \gamma(1-\beta) \\ \gamma(1-\beta) \end{pmatrix}$$

The three momentum delivered to the rocket in its rest frame is

$$\frac{dp_0}{d\tau} = 2\alpha M_0 c \frac{dt}{d\tau}\gamma(1-\beta) = M_0 a_0 \quad \longrightarrow \quad a_0 = 2\alpha c \frac{dt}{d\tau}\gamma(1-\beta)$$

Doppler shift due to photons:

$$\frac{dt}{d\tau} = \sqrt{\frac{1-\beta}{1+\beta}}$$

such that

$$a_0 = 2\alpha c \left(\frac{1-\beta}{1+\beta}\right)$$

As $\beta \to c$, $a_0 \to 0$, meaning that the first spacecraft will clearly catch it up.



### Question 2

Adopt the metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Then consider the 4-momenta in the CM frame ($c \equiv 1$):

$$\begin{aligned} \mathsf{P}'_0 &= (m_0, 0) \\ \mathsf{P}'_1 &= (E'_1, \mathbf{p}') \\ \mathsf{P}'_2 &= (E'_2, -\mathbf{p}') \end{aligned}$$

By the conservation of 4-momentum,

$$\begin{aligned} \mathsf{P}'_0 &= \mathsf{P}'_1 + \mathsf{P}'_2 \\ (\mathsf{P}'_0 - \mathsf{P}'_1)^2 &= P'^2_2 \\ -m_0^2 - m_1^2 - 2\mathsf{P}'_0 \cdot \mathsf{P}'_1 &= -m_2^2 \\ -m_0^2 - m_1^2 - 2(-m_0 E'_1) &= -m_2^2 \\ \to E'_1 &= \frac{m_0^2 + m_1^2 - m_2^2}{2m_0} \end{aligned}$$

Consider $E'_1 = \sqrt{m_1^2 + p'^2}$:

$$\begin{aligned} E_1^2 &= m_1^2 + p^2 \\ p^2 &= E_1^2 - m_1^2 \\ &= \frac{(m_0^2 + m_1^2 - m_2^2)^2 - 4m_1^2 m_0^2}{4m_0^2} \\ \to p &= \frac{1}{2m_0}\left((m_1^2 + m_2^2 - m_0^2)^2 - 4m_1^2 m_2^2\right)^{1/2} \end{aligned}$$

We are given that $p_K = 9541$ MeV and $p_\pi = 1625$ MeV, such that $\theta_0 = \tan^{-1}(0.21133\ldots) \approx 11.93°$. This is a more accurate prediction of the direction of travel of the particle.

Now,

$$\begin{aligned}
\mathsf{P}_0^2 &= (\mathsf{P}_\pi + \mathsf{P}_K)^2 \\
\mathsf{P}_0^2 &= \mathsf{P}_\pi^2 + \mathsf{P}_K^2 + 2\mathsf{P}_\pi \cdot \mathsf{P}_K \\
-m_0^2 &= -m_\pi^2 - m_K^2 + 2\left(-E_\pi E_K + \mathbf{p}_\pi \cdot \mathbf{p}_K\right) \\
m_0^2 &= m_\pi^2 + m_K^2 + 2\left(E_\pi E_K - p_\pi p_K \cos\left(\theta_\pi - \theta_K\right)\right)
\end{aligned}$$

Noting that

$$\begin{aligned}
E_\pi &= \sqrt{m_\pi^2 + p_\pi^2} \approx 1630.98 \text{ MeV} \\
E_K &= \sqrt{m_K^2 + p_K^2} \approx 9553.76 \text{ MeV}
\end{aligned}$$

it follows that the mass of the particle is given by

$$m_0 \approx 1896.9 \text{ MeV}$$

Let the proper time be $\tau$. In the lab frame, the particle takes $\gamma\tau$ to decay. Let $\ell$ be the distance travelled in the lab:

$$\gamma\tau = \frac{\ell}{v} \quad \longrightarrow \quad \tau = \frac{\ell}{\gamma v} = \frac{\ell}{c}\frac{m_0}{p_0}$$

Then,

$$p_0 = \frac{1}{\cos\theta_0}\left(p_K \cos\theta_K + p_\pi \cos\theta_\pi\right) \approx 11023.5 \text{ MeV}$$

such that

$$\tau \approx 4.998 \times 10^{-13} \text{ s}$$

In the COM frame, we have that

$$\begin{aligned}
\mathsf{P}_0' &= (m_0, 0) \\
\mathsf{P}_K' &= (E_K', p_K' \cos\theta_K', p_K' \sin\theta_K') \\
\mathsf{P}_\pi' &= (E_\pi', p_\pi' \cos\theta_\pi', p_\pi' \sin\theta_\pi')
\end{aligned}$$

By four-vector conservation in the lab frame:

$$\begin{aligned}
\mathsf{P}_0' &= \mathsf{P}_K' + \mathsf{P}_\pi' \\
\mathsf{P}_0' - \mathsf{P}_K' &= \mathsf{P}_\pi' \\
\mathsf{P}_\pi'^2 &= \mathsf{P}_0'^2 + \mathsf{P}_K'^2 - 2\mathsf{P}_0' \cdot \mathsf{P}_K' \\
m_\pi^2 &= m_0^2 + m_K^2 + 2(-E_K' m_0)
\end{aligned}$$

such that

$$E_K' = \frac{m_0^2 + m_K^2 - m_\pi^2}{2m_0} \approx 992.109 \text{ MeV}$$

Noting that $E_K'^2 = m_K^2 + p_K'^2$, it follows that $p_K' = 860.55$ MeV/c. Similarly,

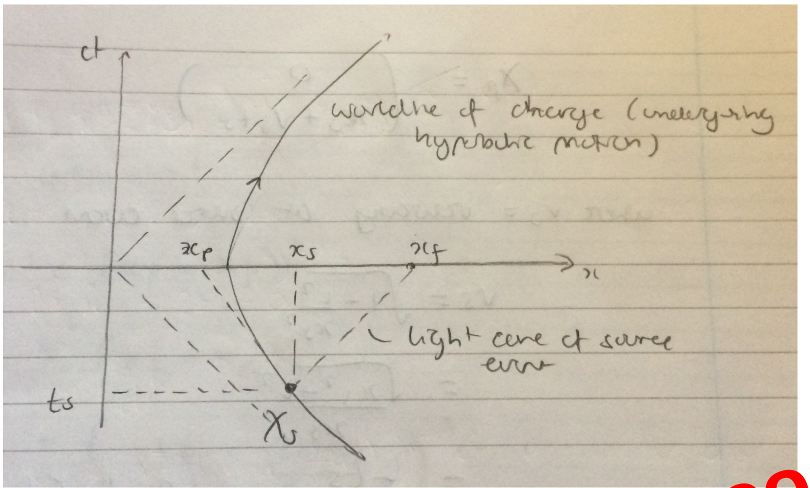
$$\begin{aligned}
E_\pi' &= \frac{m_0^2 + m_\pi^2 - m_K^2}{2m_0} \approx 871.79 \text{ MeV} \\
p_\pi' &= 860.54 \text{ MeV/c}
\end{aligned}$$

This means that the proper acceleration $\gamma$ is constant, as required.

Field event - the spacetime event at which the fields are measured.

Source event - The event on the past light-cone of the field event at which the source is located.

Projected position - The position of the charge at the field event if the charge where to continue from the source event at a constant velocity.






$$\mathsf{X}_f = (0, x_f)$$
$$\mathsf{X}_s = (t_s, x_s)$$

We require that $\mathsf{X}_{sf} = \mathsf{X}_s - \mathsf{X}_f$ is null, such that the source event lies on the past light-cone of the field event:

$$\mathsf{X}_{sf} = (t_s, x_s - x_f)$$
$$-t_s^2 + (x_s - x_f)^2 = 0$$
$$\rightarrow t_s = x_s - x_f$$

Then,

$$x_s^2 - (x_s - x_f)^2 = L^2$$

such that

$$x_s = \frac{L^2 + x_f^2}{2x_f}, \quad t_s = \frac{L^2 - x_f^2}{2x_f}$$

The projected position of the particle is given by

$$\mathsf{X}_p = (0, x_p) = (0, x_s + v_s t_s)$$

where $v_s$ is the velocity at the source event:

$$v_s = \sqrt{1 - \frac{L^2}{x_s^2}} = \frac{\sqrt{x_s^2 - L^2}}{x_s} = -\frac{t_s}{x_s}$$

We want a spinor that has associated four-vector

$$\mathsf{P} = \sqrt{E}(1,1,1,0)$$

We want to rotate $\mathbf{s}_2$ around $\mathbf{e}_z$ by $-\pi/4$ to obtain a spinor along $(1,1,0)$. So:

$$\begin{aligned}
\mathbf{s}_{\text{new}} &= \frac{1}{\sqrt{2}}\left[\begin{pmatrix} \cos\pi/8 & 0 \\ 0 & \cos\pi/8 \end{pmatrix} - i\begin{pmatrix} \sin\pi/9 & 0 \\ 0 & -\sin\pi/8 \end{pmatrix}\right]\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\pi/8} \\ e^{i\pi/8} \end{pmatrix}
\end{aligned}$$

or equivalently

$$\mathbf{s}_{\text{new}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\pi/4} \end{pmatrix}$$

Introducing the appropriate energy normalisation, this becomes

$$\mathbf{s}_E = \sqrt{\frac{E}{2}}\begin{pmatrix} 1 \\ e^{i\pi/4} \end{pmatrix}$$

In order to boost the spinor using $e^{i\rho/2}$, we need to find an expression for $\rho$.

$$\begin{aligned}
\beta = \tanh\rho &= \frac{e^{\rho} - e^{-\rho}}{e^{\rho} + e^{-\rho}} = \frac{1 - e^{-2\rho}}{1 + e^{-2\rho}} \\
\beta + \beta e^{-2\rho} &= 1 - e^{-2\rho} \\
\rightarrow e^{\rho} &= \sqrt{\frac{1+\beta}{1-\beta}} = \sqrt{\frac{1+15/17}{1-15/17}} = 4
\end{aligned}$$



In the new frame,

$$\mathbf{s}'_E = \sqrt{\frac{E}{2}}\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ e^{\pi/4} \end{pmatrix} = \sqrt{\frac{E}{2}}\begin{pmatrix} 1/2 \\ 2e^{i\pi/4} \end{pmatrix}$$

From above,

$$\mathsf{P}' = \begin{pmatrix} |a|^2 + |b|^2 \\ ab^* + a^*b \\ i(ab^* - a^*b) \\ |a|^2 - |b|^2 \end{pmatrix} = \frac{E}{2}\begin{pmatrix} 4 + 1/4 \\ e^{-i\pi/4} + e^{i\pi/4} \\ i(e^{-i\pi/4} - e^{i\pi/4}) \\ -4 + 1/4 \end{pmatrix} = E\begin{pmatrix} 17/8 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ -15/8 \end{pmatrix}$$

Thus,

$$\mathsf{P}' = E\left(\frac{17}{8}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}. -\frac{15}{8}\right)$$

AS this four-vector is null, $\mathsf{P}^0 = |\mathsf{P}|^2$, allowing us to write that:

$$\begin{aligned}
\mathsf{P}'^0 \cos\theta &= \mathsf{P}'^z \\
\theta = \cos^{-1}(\mathsf{P}^z/\mathsf{P}^0) &\approx 151.9^\circ
\end{aligned}$$

such that

$$E_{CM} = \left[2(mc^2)^2 + 2(Mc^2)^2\right]^{1/2}$$

Now,

$$v_{CM} = \frac{p_{tot}}{E_{tot}} c^2$$

where $p_{tot}$ and $E_{tot}$ are evaluated in some given frame. Choose this to be the same frame as before.

$$p_{tot}c = \sqrt{E_m^2 - m^2c^4} = \frac{c^2}{2M}(M^2 - m^2)$$
$$E_{tot} = Mc^2 + E_m = \frac{c^2}{2M}(3M^2 + m^2)$$

Thus,

$$\frac{v_{CM}}{c} = \frac{M^2 - m^2}{3M^2 + m^2}$$

The momentum of either particle in the CM is given by

$$p_{CM} = \gamma M v_{CM}$$

ie. we transform the mass $M$ initially at rest into the CM frame.

$$\gamma v_{CM} = \frac{M^2 - m^2}{3M^2 + m^2} \frac{1}{\sqrt{1 - \frac{(M^2-m^2)^2}{(3M^2+m^2)^2}}} = \frac{1}{M} \frac{M^2 - m^2}{\sqrt{8(M^2 + m^2)}}$$

such that

$$p_{CM} = M\gamma v_{CM} = \frac{M^2 - m^2}{\sqrt{8(M^2 + m^2)}}$$

as required (note $c \equiv 1$). Now, consider the case in which $M = \sqrt{5}m$. Then, these results become:



$$E_{CM} = \sqrt{2}\left[m^2c^4 + 5m^2c^4\right]^{1/2} = 2\sqrt{3}\, mc^2$$
$$\frac{v_{CM}}{c} = \frac{5m^2 - m^2}{15m^2 + m^2} = \frac{4m^2}{16m^2} = \frac{1}{4}$$
$$p_{CM} = \frac{5m^2 - m^2}{\sqrt{8(5m^2 + m^2)}} = \frac{4m^2}{\sqrt{48m^2}} = \frac{1}{\sqrt{3}} mc$$

where we have reintroduced $c$.

In the CM frame, both atoms are initially moving towards one another with the frame momentum in the CM frame. The photon is emitted, and then re-absorbed, after which the two particles travel away from one another again with the same momenta. Now, we want to find the velocities in the CM. For $c \equiv 1$,

$$\gamma v \bar{M} = \frac{1}{\sqrt{3}} m$$
$$\gamma v = \frac{1}{\sqrt{3}} \frac{m}{\bar{M}} \equiv \alpha$$
$$\frac{v^2}{1 - v^2/c^2} = \alpha^2$$
$$\to v = \frac{\alpha}{\sqrt{1 + \alpha^2}}$$

This means that

Performing the inverse transformation, we find that

$$\mathbb{T}_{\text{lab}} = \frac{1}{2}\epsilon_0 E^2 \begin{pmatrix} \gamma^2(1-\beta^2) & 0 & 0 & 0 \\ 0 & -\gamma^2(1-\beta^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\epsilon_0 E^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 01 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{T}$$

This means that $\mathbf{N} = 0$, as the stress-energy tensors are identical in both frames. The stored energy in the capacitor remains stored in the electric field between the two plates, which moves with them in the lab frame (and has the same magnitude). Energy is being absorbed on the bottom plate, and pulled out of the field on the top plate.

## Question 4

$$\phi = \frac{q}{4\pi\epsilon_0 r_{\text{sf}}}, \quad r_{\text{sf}} = |\mathbf{r}_s - \mathbf{r}_f|$$
$$\mathbf{A} = 0$$

in the rest frame of the charge. This means that the four-vector potential in its rest frame is

$$\mathsf{A}_0 = \left(\frac{q}{4\pi\epsilon_0 c}\frac{1}{r_{\text{sf}}}, \mathbf{0}\right)$$



We can find the velocity potential in any other frame by applying a Lorentz transformation with some velocity $\mathbf{v}_s$. Now, the only quantities that can realistically contribute to $\mathbf{r}_{\text{sf}}$ are $\mathsf{X}_{\text{sf}} = (ct_{\text{sf}}.\mathbf{r}_{\text{sf}})$, and the four-velocity of the source $\mathsf{U}_s = (\gamma c, \gamma \mathbf{v}_s)$

$$\mathsf{X}_{\text{sf}} \cdot \mathsf{U}_s = \gamma(-t_{\text{sf}}c^2 + \mathbf{r}_{\text{sf}} \cdot \mathbf{v}_s) = \gamma(-r_{\text{sf}}c + \mathbf{r}_{\text{sf}} \cdot \mathbf{v}_s)$$

In the rest frame of the charge, this evaluates to $-r_{\text{sf}}c$, so

$$\mathsf{A}_0 = \frac{q}{4\pi\epsilon_0}\frac{1}{(-\mathsf{X}_{\text{sf}} \cdot \mathsf{U}_s)}(1, \mathbf{0})$$

The general result is thus:

$$\mathsf{A} = \frac{q}{4\pi\epsilon_0}\frac{\mathsf{U}/c}{(-\mathsf{R} \cdot \mathsf{U})}$$

where $\mathsf{U}$ = four-velocity of the source event, $\mathsf{R} = \mathsf{X}_f - \mathsf{X}_s$ = null four-displacement from the source event to the field event. This result must be generally covariant as we have found a rank-1 tensor expression that agrees with the result in the rest frame of the charge. This must be a generally true result, with the potential in other frames being obtained by a simple Lorentz transformation.

$$\mathbf{r}_s = (a\cos\omega t_s, a\sin\omega t_s, 0)$$
$$t_s = t - r_{sf}/c$$

By definition,

$$r_{\text{sf}} = |\mathbf{r}_s - \mathbf{r}_f|$$

Now, we note that

$$|\mathbf{u}_\parallel| = u\cos\theta$$
$$|\mathbf{u}_\perp| = u\sin\theta$$

where

$$u = \sqrt{u_\parallel^2 + u_\perp^2} = \sqrt{u_x^2 + u_y^2}$$

such that

$$\gamma_{u'} u' \cos\theta' = \gamma_v \gamma_u (u\cos\theta - v)$$
$$\gamma_{u'} u' \sin\theta' = \gamma_u u \sin\theta$$

Taking the ratio of these expressions, we arrive at the desired result of

$$\tan\theta' = \frac{u\sin\theta}{\gamma_v(u\cos\theta - v)}$$

Consider the magnetic field to be orientated along $z$, so $\mathbf{B} = (0, 0, B)$. Assume the adiabatic approximation; that there are many loops of the helix for a given distance moved along $\mathbf{e}_z$ such that we can write that

$$\int dt\, \mathbf{f}_\perp = 0 \quad \longrightarrow \quad |\mathbf{p}_\perp^{(i)}| = |\mathbf{p}_\perp^{(f)}|$$

meaning that momentum is conserved in the perpendicular plane. As the Larmor radius is given by $R = |\mathbf{p}_\perp|/(eB)$, this means that it does not vary throughout the motion.

By the conservation of energy ($c = 1$),

$$E_i^2 = m^2 + p_\parallel^2 + p_\perp^2$$
$$E_f^2 = m^2 + p_\perp^2$$



Now, $E_i = 17m$ since $\gamma_i = 17$. Noting that we initially have that $p_\perp = p_\parallel$, it follows that

$$(17m)^2 = m^2 + p_\parallel^2 + p_\perp^2$$
$$289m^2 = m^2 + 2p_\perp^2$$
$$\rightarrow p_\perp = p_\parallel = 12m$$

Since $\gamma \gg 1$, $v_\parallel \backsim 1/\sqrt{2}$ throughout, such that

$$\Delta p_\parallel \approx \sqrt{2} f d \quad \longrightarrow \quad fd = \frac{p_\parallel}{\sqrt{2}} \approx \frac{12m}{\sqrt{2}}$$

Then:

$$qEd = \frac{12}{\sqrt{2}} m = 4.33 \text{ MeV}$$
$$qE = 4.33 \text{ MeV/m}$$
$$E = 4.33 \text{ MV/m}$$

At the end of the deceleration,

$$(\gamma_f m)^2 = m^2 + (12m)^2 = 145m^2$$

- $\perp$ direction:

$$p_\perp = -p_\gamma = -\frac{h}{\lambda}$$

Now,

$$\begin{aligned} E_f^2 &= \underbrace{(m_e + m_p)^2}_{M^2} c^4 + p^2 c^2 \\ &= M^2 c^4 + (p_\perp^2 + p_\perp^2)c^2 \\ &= M^2 c^4 + \left[\frac{\hbar^2}{\lambda^2} + \gamma^2 v_z^2 (m_p - m_e)^2\right] c^2 \end{aligned}$$

but

$$\gamma = \frac{1}{\sqrt{1 - v_z^2/c^2}}$$

so

$$E_f^2 = E_\gamma^2 + \frac{M^2c^4 - M^2c^2v_z^2 + v_z^2c^2(m_p - m_e)^2}{1 - v_z^2/c^2} = E_\gamma^2\left[1 + \frac{(m_e+m_p)^2c^4 - 4m_em_pv_z^2c^2}{(1 - v_z^2/c^2)E_\gamma^2}\right]$$

Thus, the minimum total energy is given by

$$E_{\text{tot}} E_\gamma^2 \left[1 + \frac{(m_e+m_p)^2c^4 - 4m_em_pv_z^2c^2}{(1 - v_z^2/c^2)E_\gamma^2}\right]^{1/2}$$

where $E_\gamma = hc/\lambda$ is the energy of the emitted photon.

Consider the field equations:

$$\begin{aligned} \mathbb{F}^{\mu\nu} &= \partial^\mu \mathsf{A}^\nu - \partial^\nu \mathsf{A}^\mu \\ \partial_\mu \mathbb{F}^{\mu\nu} &= \partial_\mu \partial^\mu \mathsf{A}^\nu - \partial^\nu(\partial_\mu \mathsf{A}^\mu) = -\mu_0 \mathsf{J}^\nu \end{aligned}$$

In the Lorentz gauge, $\partial_\mu \mathsf{A}^\mu = 0$, so

$$\partial_\mu \partial^\mu \mathsf{A}^\nu = -\mu_0 \mathsf{J}^\nu$$

In the absence of a source term,

$$\partial_\mu \partial^\mu \mathsf{A}^\nu = 0 \quad \longrightarrow \quad 0 = \begin{cases} -\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi \\ -\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla^2 \mathbf{A} \end{cases}$$

Without loss of generality, set $\phi = 0$. Then,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

such that

$$\mathbf{A} = -\int dt\, \mathbf{E} = -E_0 \int dt\, \cos(\omega t - kz)\, \mathbf{e}_y = -\frac{E_0}{\omega} \sin(\omega t - kz)\, \mathbf{e}_y$$

Thus,

$$\begin{aligned} \phi &= 0 \\ \mathbf{A} &= \left(0, -\frac{E_0}{\omega}\sin(\omega t - kz), 0\right) \end{aligned}$$

For the last case, consider the transformation:

$$\mathbf{E}_\perp = \gamma(\mathbf{E}'_\perp - \mathbf{v} \times \mathbf{B}')$$
$$\mathbf{B}_\perp = \gamma\left(\mathbf{B}_\perp + \frac{1}{c^2}\mathbf{v} \times \mathbf{E}'\right)$$

where again the primed field are in $S'$. We want to find a frame in which $\mathbf{E}_\perp = \mathbf{E} = 0$. In $S'$, $\mathbf{B}' = 0$, so

$$\mathbf{E}_\perp = \gamma\mathbf{E}'_\perp$$
$$\mathbf{B}_\perp = \frac{\gamma}{c^2}\mathbf{v} \times \mathbf{E}'_\perp$$

This means that it is not possible to find a frame in which the force is purely magnetic as there will always be an electric field. We can also consider:

$$\alpha = \mathbf{E} \cdot \mathbf{B}/c$$
$$D = B^2 - E^2/c^2$$

We have $\alpha = 0$, so $D = -E^2/c^2 < 0$ the magnetic field can be taken to zero, but not the electric field (as $D$ is sign definite).