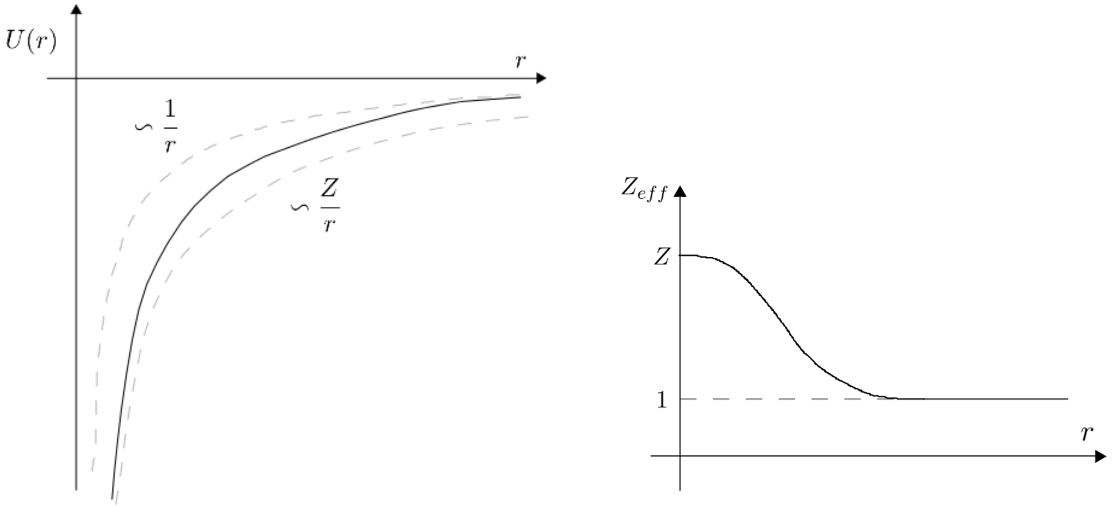

(a) (b)

Figure 2.1: (a) An approximate form of the central potential $U(r)$ (b) An approximate form for the effective charge Z_{eff}

where $U(r) \backsim 1/r$.

Finding $U(r)$ exactly is a task for recursive numerical methods, such as the Hartee method. This involves the following steps:

1. Guess a form of the potential $U(r)$
2. Solve for the single electron wavefunctions $\phi_i(\mathbf{r})$
3. Find the charge density $\rho(\mathbf{r}) = -\sum_i e\,|\phi_i(\mathbf{r})|^2$
4. Find the resultant potential $U(r)'$ from the electric field $\mathbf{E}$

One can then iterate steps 2 to 4 until a self-consistent solution is found. This can be extended further by the Hartee-Fock method, which includes the effects of Pauli exclusion principle in the algorithm. As a result, it generates a central potential that gives rise to fully anti-symmetrised spatial wavefunctions.

2.1.2 Electronic Configurations

Each electron within an multiple electron atom has a particular n and ℓ associated with it; m_ℓ makes no contribution to the energy. Electronic configurations are a way of specifying these quantum numbers, notated as below.

The values of ℓ are denoted using spectroscopic notation. That is, we used the letters $s, p, d, f, \dots$ to refer to $\ell = 0, 1, 2, 3, \dots$ and so on. There is no particular logic behind why this is the case; it is simply a notational convention, and one that has to be learnt.

Each *energy orbital* is degenerate in n and ℓ; there are $2n^2$ states for each value of n (a consequence of Pauli exclusion), and $2(2\ell + 1)$ for each value of ℓ_i (factor of two due to spin degeneracy). It is important to remember that $0 \leq \ell_i \leq n$, and that the sum of the number of electrons in all of the orbitals is equal to the atomic number Z (in the case of

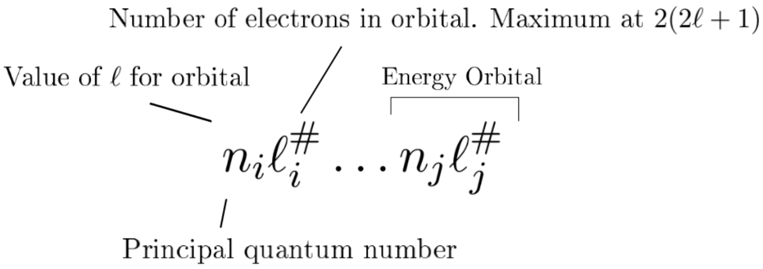

Figure 2.2: Notation for electronic configurations

neutral atoms). These can be useful consistency checks when constructing configurations.

The energy orbitals with the lowest energy will fill up first, and for all elements up for Argon (Ar), this corresponds to the orbitals with the lowest n. After Ar, we find that

$$\begin{aligned}\text{Ar}:&1s^2 2s^2 2p^6 3s^2 3p^6\\ \text{K}:&1s^2 2s^2 2p^6 3s^2 3p^6\quad\ 4s\\ \text{Ca}:&1s^2 2s^2 2p^6 3s^2 3p^6\quad\ 4s^2\\ \text{Sc}:&1s^2 2s^2 2p^6 3s^2 3p^6 3d\,4s^2\end{aligned}$$

and so on. Logically, we would expect the next orbital to be filled in Potassium (K) to be the $3d$ orbital. However, the fact that it has $\ell = 2$ means that - on average - the electron is localised further away from the nucleus, and so is partially screened from the nuclear charge. This causes it to be higher in energy than the $4s$ state, an so it does not fill up next. For higher elements, the states become significantly more complicated to deduce, though it is not particularly crucial to be able to write down their electronic configurations; rather, one should be able to extract information from a given electronic configuration.

We note that full orbitals make no contribution to the overall angular momentum; this is because they must have spherically symmetric electrostatic potentials, forcing $\ell = 0$ (spherically symmetric states cannot have a non-zero value of ℓ, as to do so would define a special spatial direction, creating a contradiction).

2.1.3 Alkali Atoms

The Akali metals are found in the first group of the periodic table, and have a single valence electron that is responsible for all the system dynamics. As the electron core consists of full shells, it makes no contribution to ℓ and s, as above. For example, consider Sodium (Na, $Z = 11$):

$$\text{Na}:\underbrace{1s^2 2s^2 2p^6}_{\text{core}}\quad\underbrace{n\ell}_{\text{valence electron}} \tag{2.6}$$

The core has the same electronic configuration as Neon (Ne), and one will often see the configuration in the first bracket simply replaced by '[Ne]' in shorthand notation. This makes it explicit that Na is simply a Ne electron core with an extra valence electron.

This valence electron thus moves in a hydrogenic potential, as it is shielded by the inner electrons most of the time. This means that for large ℓ, we can write that $Z_{eff} \backsim Z-(Z-1) = 1$. However, this approximation begins to break down for lower ℓ, as the electron has a higher probability of being found in locations close to the nucleus. We take account of

3. *Radiation*

In this chapter, we will examine the theory behind transitions and radiation within atoms, including

- Radiative Transitions
- Inner Shell Transitions
- Thermal Radiation

The previous two chapters was focussed mainly on outlining the various energy levels that are present in the atom, given the perturbative level being considered. We are now going to look at what happens when there are radiative transitions between these levels. This is an important aspect to round off our examination of the atom, as it is usually through observing these transitions that we can make deductions about atomic structure.

The energies of the X-ray transitions are then given by the difference in energy between these two levels. For a transition $n \mapsto m$, this is given by:

$$\Delta E_{nm} = \frac{1}{2}\mu(\alpha c)^2 \left(\frac{(Z-\sigma_n)^2}{n^2} - \frac{(Z-\sigma_m)^2}{m^2} \right) \tag{3.27}$$

We can observe these transitions using (for example) X-ray spectroscopy, involving bombarding an sample of a particular element with high-energy electrons emitted from an X-ray tube. This creates a continuous spectrum due to the *Bremsstrahlung* or *breaking* radiation due to the deceleration of the tube electrons as they interact with the atomic nuclei. There is a low-wavelength cut-off that occurs where all the energy of the electron is emitted as breaking radiation, bringing an end to the continuum. This means that the Bremsstrahlung continuum is actually characteristic of the tube voltage, rather than the specific element under study.

Superimposed on this continuum we have the characteristic X-rays emissions that correspond to inner shell transitions, with energies given by (3.27). These usually come in various groups that correspond to transitions to $n = 1, 2, 3, \dots$, known as the K-series, L-series and M-series. One usually only observes three such series due to the fact that transitions to $n = 4$ are often not energetic enough to create X-rays upon emission. Note that the term *primary electrons* is sometimes used to refer to the electrons that are ionised due to incoming radiation. The difference between their kinetic energy and the energy of the incoming radiation gives the binding energies of the various energy levels. Pay attention to selection rules when finding the characteristic X-rays; some transitions are not allowed!

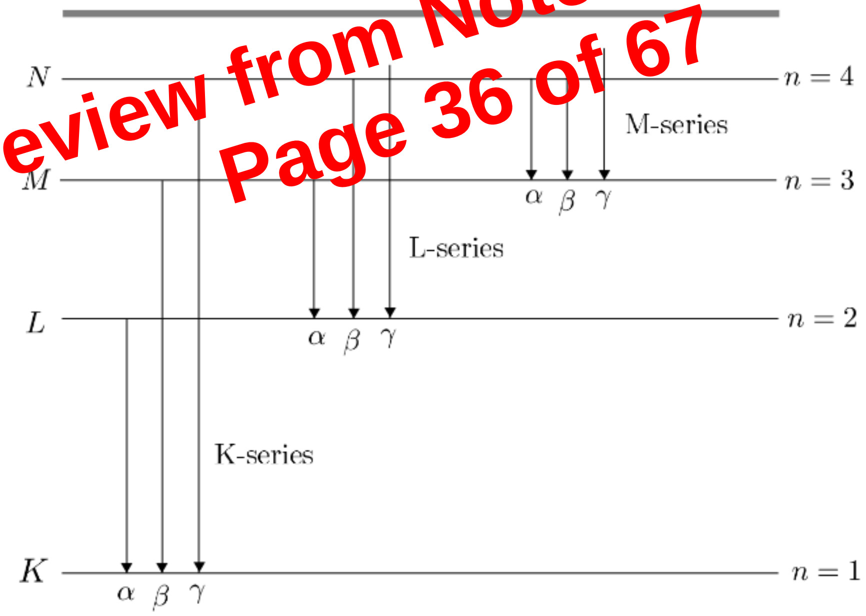

Figure 3.2: An energy level diagram showing the various X-ray series, as well as the individual lines (e.g. $K_\alpha, K_\beta, \dots$)

A X-ray tube operates with a tungsten target ($Z = 74$). As the tube voltage is increased, three groups of lines appear. The first appears at a tube voltage of $\backsim 2.5$ kV, with wavelengths around 0.65 nm. The second appears at $\backsim 12$ kV, the wavelengths being around 0.13 nm. Estimate the tube voltage at which the final group appears, and the wavelength of

We thus obtain the differential equation

$$u = \frac{1}{3}T\frac{du}{dT} - \frac{u}{3} \longrightarrow 4u = T\frac{du}{dT} \tag{3.45}$$

that we can solve to obtain $u \propto T^4$. This means that *incident power per unit area* is given by

$$\boxed{P = \frac{1}{4}uc = \sigma T^4} \tag{3.46}$$

for some constant of proportionality σ that is known as the *Stefan-Boltzmann constant*. However, in order to find an exact expression for σ, we need to investigate how our energy is distributed in frequency space.

Plank distribution

The *spectral energy density* $\rho(\omega)$ is the energy density per unit volume, in the frequency range $[\omega, \omega + d\omega]$. The photons in our gas have two possible polarisation states ($s = \pm 1$), meaning that the density of states for such a system is given by

$$g(k)d^3\mathbf{k} = \underbrace{2}_{\text{polarisation states}} \frac{V}{(2\pi)^3}4\pi k^2 dk = \frac{Vk^2}{\pi^2}dk \tag{3.47}$$

Using the dispersion relation $\omega = ck$, this can be written as

$$g(\omega)d\omega = \frac{V}{\pi^2 c^3}\omega^2 d\omega \tag{3.48}$$

Assuming that the number of photons is large enough to approximate them as having a continuous spectrum, we can adopt Bose-Einstein statistics to write the mean occupation number for a particular state with energy $\hbar\omega$ as

$$\bar{n}_i = \frac{1}{e^{\beta\hbar\omega} - 1} \tag{3.49}$$

where we have defined $\beta \equiv k_B T$. The spectral energy density is thus given by

$$\boxed{\rho(\omega) = \bar{n}_i g(\omega)\hbar\omega = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\beta\hbar\omega} - 1}} \tag{3.50}$$

This is known as the *Plank distribution*. We can then use this to find an expression for the Stefan-Boltzmann constant:

$$u = \int d\omega\, \rho(\omega) = \frac{\hbar}{\pi^2 c^3}\int_0^\infty d\omega\, \frac{\omega^3}{e^{\beta\hbar\omega} - 1} = \frac{\hbar}{\pi^2 c^3}\frac{1}{(\hbar\beta)^4}\underbrace{\int_0^\infty dx\, \frac{x^3}{e^x - 1}}_{\pi^4/15} \tag{3.51}$$

where we have made the substitution that $x = \beta\hbar\omega$. By comparison with (3.41), it is clear that the Stefan-Boltzmann constant can be expressed as

$$\boxed{\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2} = 5.67 \times 10^{-8}\ \mathrm{W\,m^{-2}K^{-4}}} \tag{3.52}$$

By letting $\omega = 2\pi c/\lambda$, we can also write the spectral energy density in wavelength space as

$$\rho(\lambda) = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{\beta hc/\lambda} - 1} \tag{3.53}$$

Since the electron angular momentum $\mathbf{L}$ precesses rapidly around the internuclear axis, the only non-zero time-averaged component of $\mathbf{L}$ is L_z, meaning that

$$\langle\psi_e|\,\mathbf{N}\cdot\mathbf{L}\,|\psi\rangle = 0 \quad \text{and} \quad \langle\psi_e|\,\mathbf{L}^2\,|\psi_e\rangle = \langle\psi_e|\,L_z^2\,|\psi_e\rangle = \Lambda^2\hbar^2 \tag{4.25}$$

This means that the rotational energy term becomes

$$\frac{\langle\psi_e|\,\mathbf{N}^2\,|\psi_e\rangle}{2\mu R^2} = \frac{\hbar^2}{2\mu R^2}\left[K(K+1)-\Lambda^2\right] \tag{4.26}$$

Now, since $E_e(R)$ is spherically symmetric, we can write the molecular wavefunction in the form

$$\langle\mathbf{x}|\psi_n\rangle = \psi_n(\mathbf{R}) = \psi_{\text{vib}}(R)\psi_{\text{rot}}(\Theta,\Phi) \tag{4.27}$$

where Θ and Φ are the angular coordinates associated with lab coordinate system, rather than the body centred coordinate system used to solve (4.19). In a similar fashion to section 1.1.1, we can separate the angular and radial equations, noting now that

$$K^2\psi_{\text{rot}}(\Theta,\Phi) = K(K+1)\hbar^2\,\psi_{\text{rot}}(\Theta,\Phi) \tag{4.28}$$

$$K_z\psi_{\text{rot}}(\Theta,\Phi) = M_K\hbar\,\psi_{\text{rot}}(\Theta,\Phi) \tag{4.29}$$

It thus follows that we can write

$$\left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial R^2} + V_{\text{eff}}(R)\right]\psi_{\text{vib}}(R) = E\psi_{\text{vib}}(R) \tag{4.30}$$

where

$$\boxed{V_{\text{eff}}(R) = E_e(R) + \frac{\left[K(K+1)-\Lambda^2\right]\hbar^2}{2\mu R^2}} \tag{4.31}$$

is the *effective potential* for this one-dimensional Schrödinger equation.

The Effective Potential

For any bound state, we can expand $E_e(R)$ around its minimum value R_0:

$$E_e(R) = E_e(R_0) + \cancel{\left.\frac{dE_e}{dR}\right|_{R_0}(R-R_0)} + \frac{1}{2!}\left.\frac{d^2E_e}{dR^2}\right|_{R_0}(R-R_0)^2 + \dots \tag{4.32}$$

The second term is zero by definition of the minimum. Letting

$$K_s = \left.\frac{d^2E_e}{dR^2}\right|_{R_0} \quad \text{and} \quad B_r = \frac{\hbar^2}{2\mu R_0^2} = \frac{\hbar^2}{2I_M} \tag{4.33}$$

we can approximate the value of the effective potential around the equilibrium value R_0 as

$$V_{\text{eff}}(R) \backsim E_e(R_0) + B_rK(K+1) + \frac{1}{2}K_s(R-R_0)^2 \tag{4.34}$$

where we have absorbed the $\Lambda^2\hbar^2/2\mu R_0^2$ term into $E_e(R_0)$. It thus becomes clear that V_{eff} has the form of a harmonic potential raised by a constant energy $E_e(R_0)+B_rK(K+1)\hbar^2$. Thus, the eigenvalue solution to (4.30) is given by

$$\boxed{E = E_e(R_0) + \left(\nu+\frac{1}{2}\right)\hbar\omega_{\text{vib}} + B_rK(K+1), \quad \omega_{\text{vib}} = \left(\frac{K_s}{\mu}\right)^{1/2}} \tag{4.35}$$

can evaluate the electronic integral around $\mathbf{R} = \mathbf{R}_0$. Then:

$$D_{21} \backsim \underbrace{\int d\tau_e \, \psi_e^*(\mathbf{r}_i, \mathbf{R}_0)\mathbf{D}\psi_{e'}(\mathbf{r}_i, \mathbf{R}_0)}_{\text{electronic, } I_e} \underbrace{\int dR \, \psi_{\text{vib}}^*(R)\psi_{\text{vib}'}}_{\text{vibrational, } I_{\text{vib}}} \times \underbrace{\int d\Theta d\Phi \, \sin\Theta \, \psi_{\text{rot}}^*(\Theta,\Phi)\mathbf{D}\cdot\widehat{\mathbf{E}}\psi_{\text{rot}'}(\Theta,\Phi)}_{\text{rotational, } I_{\text{rot}}} \tag{4.45}$$

We can use this expression to motivate the selection rules associated with molecular transitions. Evidently, if any part of this integral evaluates to zero, then the transition is forbidden under the electric dipole approximation.

Transitions with no change of electronic states

Transitions of this type are only involved in heteronuclear molecules, since for homonuclear molecules the I_e is zero as $\mathbf{D}_n = 0$ as $\mathbf{R}_1 = -\mathbf{R}_2$, and $\mathbf{D}_e$ makes no contribution to the integral as it is of definite parity.

If the Born-Oppenheimer approximation holds rigorously, then transitions involving a change in both the rotational and vibrational quantum numbers is forbidden, since within a given electronic state, the vibrational wavefunctions are zero, forcing $I_{\text{vib}} = 0$. However, in practise, I_e is not independent of $\mathbf{R}$; if on expands the integral around $\mathbf{R} = \mathbf{R}_0$, an extra term is introduced. This gives rise to the selection rules:

$$\boxed{\begin{aligned} &|K - K'| \leq 1 \quad \text{except } |K - K'| \neq 0 \text{ for } \Lambda = \Lambda' = 0 && (4.46)\\ &|\nu - \nu'| = 1 && (4.47) \end{aligned}}$$

Transitions with a change of electronic state

If the transition involves a change in electronic state, then I_e can be nonzero for both homonuclear and heteronuclear molecules, as the wavefunctions $\psi_e(\mathbf{r}_i, \mathbf{R}_0)$ and $\psi_{e'}(\mathbf{r}_i, \mathbf{R}_0)$ will be distinct. In this case, we retain the selection rules above, except now we have the additional rules:

$$\boxed{\begin{aligned} |S - S'| &= 0 && (4.48)\\ |\Lambda - \Lambda'| &\leq 1 && (4.49)\\ \Sigma^{\pm} &\rightarrow \Sigma^{\pm} && (4.50)\\ g &\rightarrow u && (4.51) \end{aligned}}$$

where the last two apply to Σ states and homonuclear molecules respectively. It is clear that the symmetry associated with Σ cannot change under a transition, while homonuclear molecules must undergo a parity inversion.

4.2.3 Rabi Oscillations

> *You can have Rabbi oscillations, but that requires two Jews and a see-saw*
> \- Professor Simon Hooker

The term *Rabi oscillations* (pronounced "rah-bee") is used to refer to the oscillation of the atomic population between the two levels of a two-level system at some characteristic frequency. Let us label our eigenstates of our two-level system by $|1\rangle$ and $|2\rangle$. Suppose that monochromatic light of amplitude $\mathbf{E}_0$ and frequency ω is incident on the atom, such that we can write a time-dependent perturbation

$$\delta H(t) = e\mathbf{x} \cdot \mathbf{E}_0 \cos \omega t \tag{4.64}$$

We can the write the overall state of our system as

$$|\psi, t\rangle = c_1 e^{-iE_1 t/\hbar} |1\rangle + c_2 e^{-iE_2 t/\hbar} |2\rangle \tag{4.65}$$

where the time evolution of the coefficients amplitudes c_1 and c_2 are given by (3.4)

$$\dot{c}_2 = -\frac{i}{\hbar} \langle 2| \delta H |1\rangle c_1 e^{i\omega_0 t}, \quad \dot{c}_1 = -\frac{i}{\hbar} \langle 1| \delta H |2\rangle c_2 e^{-i\omega_0 t} \tag{4.66}$$

where $\omega_0 = \omega_{21}$. Let δH_{ij} represent the entries of the matrix corresponding to the perturbation δH. If we assume δH is real - which is a valid thing to do as it represents a measurable quantity - then it is clear that the matrix must be Hermitian, such that $\delta H_{12}^* = \delta H_{21} \longrightarrow \delta H_{12} = \delta H_{21}$. The fact that $\delta H_{11} = \delta H_{22} = 0$ follows from the fact that $\mathbf{x}$ has odd parity, while our basis states $|1\rangle$ and $|2\rangle$ are of even parity, forcing the matrix elements to be zero. We can thus write our matrix element as

$$\delta H_{12} = \frac{\hbar\Omega}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) \quad \text{for} \quad \boxed{\Omega = \frac{1}{\hbar} \langle 1| e\mathbf{x} \cdot \mathbf{E}_0 |2\rangle} \tag{4.67}$$

The quantity Ω is the Rabi frequency that is associated with the oscillations; it is clear that this increases with the strength of the electric field. This allows us to write the time evolution of c_2 as

$$\dot{c}_2 = -\frac{i\Omega}{2} c_1 \left(e^{i(\omega+\omega_0)t} + e^{i(\omega_0-\omega)t}\right) \tag{4.68}$$

We now make what is known as the *rotating wave approximation*, by assuming that the frequency of the electric field perturbation is close to the atomic resonance ω_0. That is, $\omega = \omega_0 + \delta\omega$, where $\delta\omega \ll \omega_0$. Then, we have that

$$|\omega - \omega_0| \ll \omega_0 \quad \text{and} \quad \omega + \omega_0 \backsim 2\omega_0 \tag{4.69}$$

This means that we can ignore the fast oscillating term $\omega + \omega_0$ in comparison to $\omega - \omega_0$, as the oscillations associated with the former will average to zero over the time scales associated with the latter. Thus, neglecting the $\omega + \omega_0$ terms in (4.66), we can write

$$\dot{c}_2 = -\frac{i\Omega}{2} c_1 e^{-i\delta\omega t}, \quad \dot{c}_1 = -\frac{i\Omega}{2} c_2 e^{i\delta\omega t} \tag{4.70}$$

Recalling the fact that both the exponential factor and the coefficients are time dependent, we can solve this coupled system to obtain a second order differential equation for c_2 (or alternatively c_1):

$$\ddot{c}_2 - i\delta\omega\, \dot{c}_2 + \left|\frac{\Omega}{2}\right|^2 c_2 = 0 \tag{4.71}$$

4.3.3 Line Broadening and the Einstein Coefficients

Now that we have made the admission that our transition frequency is not in fact a delta function, let us look at the effect that this has on our Einstein coefficients. Once again, we look at the three different processes, assuming that they are all homogeneously broadened:

- Spontaneous emission - The rate per unit volume, in the frequency range $[\omega, \omega + d\omega]$, at which this spontaneous decay occurs can be written as $n_2 A_{21} g_A(\omega - \omega_0) d\omega$, where $g_A(\omega - \omega_0)$ is the lineshape for spontaneous emission

- Absorption - The rate per unit volume, in the frequency range $[\omega, \omega + d\omega]$, at which this excitation occurs is given by $n_1 B_{12} g_B(\omega - \omega_0) \rho(\omega_{21}) d\omega$, where $g_B(\omega - \omega_0)$ is the lineshape for absorption.

- Stimulated emission - The rate at which this occurs is given by $n_2 B_{21} g_{B'}(\omega - \omega_0) \rho(\omega_{21}) d\omega$, where $\rho(\omega_{21})$ is as before, where $g_{B'}(\omega - \omega_0)$ is the lineshape for absorption.

Note that we have assumed that the lineshapes for each of the transitions are distinct. But how are they related? We can follow an identical method used in section 4.2.1 for a system in thermal equilibrium to show that

$$g_1 B_{12} g_B(\omega - \omega_0) = g_2 B_{21} g_{B'}(\omega - \omega_0) \tag{4.89}$$

$$A_{21} g_A(\omega - \omega_{)}) = \frac{\hbar \omega_0^3}{\pi^2 c^3} B_{21} g_{B'}(\omega - \omega_0) \tag{4.90}$$

As these expressions must be hold for all possible functions, it immediately follows from comparison with (4.55) that $g_A(\omega - \omega_0) = g_B(\omega - \omega_0) = g_{B'}(\omega - \omega_0)$. Thus, all of the transition types have a lineshape that is equal to some homogeneous lineshape function $g_H(\omega - \omega_0)$.

For narrow band radiation, the gain cross-section varies slowly over the spectral width of the radiation, and so $I(\omega)$ acts as $I(\omega) = I_T\delta(\omega - \omega_L)$, where ω_L is the central frequency of the beam, and I_T is the *total intensity* defined as

$$I_T(z) = \int_0^\infty d\omega\, I(\omega, z) \tag{4.106}$$

It follows that

$$\boxed{\frac{dn_2}{dt} = R_2 - n^*\sigma_{21}(\omega_L - \omega_0)\frac{I_T}{\hbar\omega_L} + \dots} \tag{4.107}$$

We can think of n^* as being the effective number density of the inverted atoms, and $\sigma_{21}(\omega_L - \omega_0)$ as their effective cross-sectional area. Note that $I_T/\hbar\omega_L$ is the incident photon flux.

We have already derived (4.95) that describes the growth of each spectral component of a beam. To describe the rate of growth of a beam of finite spectral width, we integrate both sides over the bandwidth of the beam:

$$\int_0^\infty d\omega\, \frac{\partial I}{\partial z} = \frac{\partial}{\partial z}\int_0^\infty d\omega\, I(\omega, z) = \int_0^\infty d\omega\, n^*\sigma_{21}(\omega - \omega_0)I(\omega, z) \tag{4.108}$$

For narrow band radiation, we once again assume Delta-function behaviour. Using (4.106), we find that

$$\boxed{\frac{dI_T}{dz} = n^*\sigma_{21}(\omega_L - \omega_0)I_T} \tag{4.109}$$

4.4.2 Gain Saturation

From now on, we shall adopt the notation $I = I_T$ unless stated otherwise. Let us now look at the level populations of a laser operating under steady-state conditions in the presence of an intense, narrow-band radiation. By analogy to (4.107), we can write the rate equations as

$$\frac{dn_2}{dt} = R_2 - n^*\sigma_{21}(\omega_L - \omega_0)\frac{I}{\hbar\omega_L} - \frac{n_2}{\tau_2} \tag{4.110}$$

$$\frac{dn_1}{dt} = R_1 + n^*\sigma_{21}(\omega_L - \omega_0)\frac{I}{\hbar\omega_L} + n_2A_{21} - \frac{n_1}{\tau_1} \tag{4.111}$$

We assume that R_1 and R_2 are constant, independent of n_1 and n_2, and include both direct (collision excitation) and indirect (radiative pumping, or non-radiative cascades) processes. Taking the steady state solutions of these equations, and eliminating n_1 and n_2, we obtain

$$n^* = \frac{R_2\tau_2[1 - (g_2/g_1)A_{21}\tau_1] - (g_2/g_1)R_1\tau_1}{1 + \sigma_{21}\frac{I_T}{\hbar\omega_L}[\tau_2 + (g_2/g_1)\tau_1 - (g_2/g_1)A_{21}\tau_1\tau_2]} \tag{4.112}$$

Inspecting the above expression, it is clear that the denominator is equal to unity for $I = 0$, meaning that the numerator must be the population inversion produced by the pumping in the absence of the beam. We can thus re-write the above equation as

$$\boxed{n^*(I) = \frac{n^*(0)}{1 + I/I_S}} \tag{4.113}$$

We introduce the parameters

$$\boxed{I_S = \frac{\hbar\omega_L}{\sigma_{21}\tau_R}, \quad \tau_R = \tau_2 + \frac{g_2}{g_1}\tau_1[1 - A_{21}\tau_2]} \tag{4.114}$$

4.4.3 Beam Growth

Our intensity equation (4.109) states that

$$\frac{dI}{dz} = \alpha_I I = \frac{\alpha_{21}}{1 + I/I_s} I \tag{4.117}$$

which can be integrated to give

$$\boxed{\log\left[\frac{I(z)}{I_0}\right] + \frac{I(z) - I_0}{I_s} = \alpha_{21} z} \tag{4.118}$$

This equation is transcendental, and does not have a general analytical solution. However, we have the two limits that

$$I(z) = I_0 e^{\alpha_{21} z} \quad \text{for } I(z) \ll I_s \text{ (weak beam)} \tag{4.119}$$

$$I(z) = I_0 + \alpha_{21} I_s z \quad \text{for } I_0 \gg I_s \text{ (heavy saturation)} \tag{4.120}$$

That is, at low intensities, the beam grows exponentially with distance, but once the laser transition becomes heavily saturated, the intensity grows linearly. Note that we can approximate $\tau_R \backsim \tau_2$ for predominantly radiative transitions ($A_{21} \backsim \tau_2^{-1}$), and that $n^* \backsim n_2 \backsim R_2\tau_2$ for a short groundstate lifetime with upper level pumping rate R_2. This allows us to obtain approximate expressions for I_s and α_{21}.