1.2 Partial Wave Theory

An alternative approach to quantum mechanical scattering is that of *partial wave analysis*. As these (elastic) scattering events have azimuthal symmetry, the orbital angular momentum of the particles must be conserved. This means that we can decompose the incoming and outgoing waves as a superposition of states of well-defined angular momentum:

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1)J_\ell(kr)P_\ell(\cos\theta) \tag{1.47}$$

where $J_\ell(kr)$ are the Bessel functions of the first kind, and $P_\ell(\cos\theta)$ are the familiar Legendre polynomials

$$P_0 = 1, \quad P_1 = \cos\theta, \quad P_2 = \frac{1}{2}(3\cos^2\theta - 1) \tag{1.48}$$

The radial Schrödinger equation then reads

$$\frac{d^2u_\ell}{dr^2} - \frac{\ell(\ell+1)}{r^2}u_\ell + \frac{2m}{\hbar^2}[E - V(r)]u_\ell = 0, \quad u_\ell(r) = rJ_\ell(kr) \tag{1.49}$$

As before, the aim is to solve this equation for a given scattering potential $V(r)$.

1.2.1 Partial Wave Scattering

As before, assume a localised potential ($Vr \to 0$ as $r \to \infty$). Far away from the scattering centre, the solutions must reduce approximately to plane waves. For this, we use the large argument limit of the Bessel functions

$$\lim_{r\to\infty} J_\ell(kr) = \frac{\sin\left(kr - \frac{\ell\pi}{2}\right)}{kr} = \begin{cases} \frac{\sin(kr)}{kr} & \text{for } \ell \text{ odd} \\ \frac{\cos(kr)}{kr} & \text{for } \ell \text{ even} \end{cases} \tag{1.50}$$

Adopting a solution of the form

$$\langle \mathbf{x}|\psi\rangle = \sum_{\ell=0}^{\infty} c_\ell J_\ell(kr)P_\ell(\cos\theta) \tag{1.51}$$

for constants c_ℓ, it can be shown that the associated scattering amplitude and differential cross section are given by

$$\boxed{\begin{aligned} f(\theta) &= \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)e^{i\delta_\ell}\sin\delta_\ell & (1.52) \\ \sigma &= \frac{4\pi}{k^2}\sum_{\ell}(2\ell+1)\sin^2\delta_\ell & (1.53) \end{aligned}}$$

δ_ℓ is the phase shift in the plane wave solution due to the potential for a given value of ℓ, which can be obtained through solving the radial Schrödinger equation with appropriate boundary conditions. We observe that a partial wave will have a small (large) contribution to the cross section if the phase shift is close to $n\pi$ (close to $\left(n + \frac{1}{2}\right)\pi$) for some integer n.

Consider (1.53). For elastic scattering, the phase shifts δ_ℓ must be real, meaning that

$$\sigma_\ell \leq \frac{4\pi}{k^2}(2\ell+1) \tag{1.54}$$

This is often referred to as the *unitarity limit*, which places an upper bound on the scattering cross sections for a given partial wave.

2. *Nuclear Physics*

This chapter aims to cover the basic concepts of nuclear physics, including:

- The Structure of the Nucleus
- Decay Modes
- Nuclear Fusion and Fission

Now that we have a relatively in-depth understanding of scattering theory, we can now look at some of its consequences when applied to the structure of the nucleus. We shall also investigate the physics behind the various decay processes that the nucleus can undergo.

The Asymmetry Term

We can derive the explicit form for the asymmetry term by considering the nucleons to be a degenerate Fermionic gas, with mean occupation numbers

$$\bar{n}_i = \frac{1}{e^{\beta(E-\mu)} - 1} \tag{2.6}$$

For these calculations, we shall assume that $m_p \approx m_n \equiv m$. Assuming that the nucleons are non-relativistic (a pretty safe assumption), such that we can adopt the usual energy dispersion relation

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \tag{2.7}$$

such that the density of states becomes

$$g(E)dE = \frac{(2s+1)Vm^{3/2}}{\sqrt{2}\pi^2\hbar^3} E^{1/2} dE \tag{2.8}$$

Then, we can determine E_F as the analytic solution to some equation for N, the number of nucleons of a specific species within the system, remembering that $\bar{n}_i$ behaves as a step-function for $E \ll E_F$ (it is easy to show that this is satisfied in the nucleus):

$$N = \sum_n \bar{n}_i = \int_0^{E_F} dE\, g(E) = \frac{(2s+1)Vm^{3/2}}{\sqrt{2}\pi^2\hbar^3} \frac{2}{3} E_F^{3/2} \tag{2.9}$$

Rearranging, we find that the Fermi energy is given by

$$E_F = \left(\frac{6\pi^2 n}{2s+1}\right)^{2/3} \frac{\hbar^2}{2m} \tag{2.10}$$

It follows that the mean energy is given by

$$\langle E \rangle = \sum_i \bar{n}_i E = \int_0^{E_F} dE\, g(E) E = \frac{3}{5} N E_F \tag{2.11}$$

For the entire nucleus, the mean energy is the sum of the mean energies of each species. Consider the case where $N = Z = A/2$. Then, the Fermi energy for both species is given by

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} = \frac{\hbar^2}{2mr_0^2}\left(\frac{9\pi}{8}\right)^{2/3} \tag{2.12}$$

where we have used the fact that $V = (4\pi/3)r^3 = (4\pi/3)r_0^3 A$ as per (2.3). In the case that $N \approx Z$, we have that

$$\langle E \rangle = \langle E_N \rangle + \langle E_Z \rangle = \frac{3}{5} E_F \frac{N^{5/3} + Z^{5/3}}{(A/2)^{2/3}} \tag{2.13}$$

Let $\delta = (Z - N)/A$, such that $N = A(1-\delta)/2$ and $Z = A(1+\delta)/2$. As δ is small, we can expand $(1 \pm x)^{5/3} \approx 1 \pm \frac{5}{3}x + \frac{1}{2!}\frac{5}{3}\left(\frac{5}{3} - 1\right)x^2$. Collecting like terms, we find that the asymmetry term must be given by

$$E_{\text{asymmetry}} = \frac{1}{3} E_F \frac{(N-Z)^2}{A} \tag{2.14}$$

Evaluating the pre-factor, we find that $E_F/3 \approx 11.2$ MeV, just under half of the value observed experimentally in table 2.1. This is due to the fact that we have not taken into account the effect that the occupation symmetry has on the potential energy of the nucleus.

2.3.2 Fission

Fission is the splitting of a nucleus into two daughter nuclei. This occurs when the nucleus is massive enough that the barrier provided the the electrostatic interaction is small. We can model this process as the pulling apart of two charged spheres. During their separation, the charges move further apart, decreasing the magnitude of the Coulomb term in the SEMF (2.5). However, the surface area must increase, resulting in an increase in surface energy. We can thus approximate the barrier height at the point at which the daughter nuclei have *just* separated by

$$E_{\text{barrier}} = \frac{\text{Coulomb Energy}}{\text{Surface energy}} = \frac{\hbar c \alpha_{EM} (Z/2)^2}{2r_0 (A/2)^{1/3}} \tag{2.47}$$

where $Z/2$ and $A/2$ are the atomic number and mass number respectively of the daughter nuclei (assumed symmetric split). For $A \approx 200$, the barrier is small enough that tunnelling can occur, the rate of which dictates the rate of *spontaneous fission.*

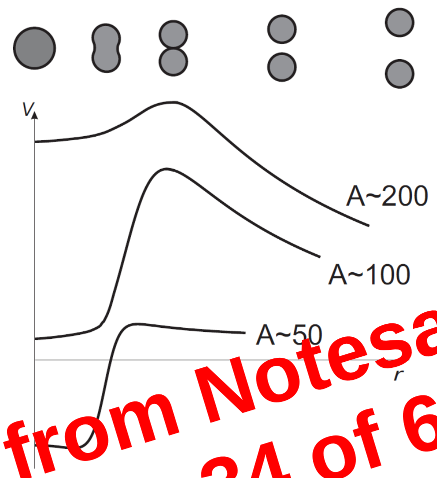

Figure 2.4: The shape of the potential barrier against fission for different values of the mass number A. Above, we have indicated a visual representation of their separation

However, we can also induce fission. Suppose that a thermal neutron ($\backsim$ eV) is incident on a nucleus with odd neutron number N. This will lead to an even N nucleus. This will liberate the normal binding energy, plus the energy from the δ pairing term as the even N state is lower in energy than the odd N state. For some elements, this liberates enough energy to give a large enough probability of tunnelling through the Coulomb barrier, and thus inducing fission. Apart from the energy liberated and the daughter nuclei, the fission also produces - on average - some extra neutrons. These may either be absorbed by non-fissile material, or go onto induce further fissions, leading to a chain reaction.

The canonical examples of fissile materials are the uranium isotopes ^{238}U and ^{235}U. ^{238}U does not undergo fission with thermal neutrons (only possible for $\backsim$ MeV neutrons), as bombardment with these neutrons leads to a γ decay process:

$$^{238}\text{U} + n \longrightarrow ^{238}\text{U}^* \longrightarrow ^{238}\text{U} + \gamma \tag{2.48}$$

However, ^{235}U is fissile as the transition from an even-odd nucleus ($^{235}_{92}$U) to an even-even nucleus ($^{236}_{92}$U) releases the pairing energy, allowing fission to occur:

$$^{235}\text{U} + n \longrightarrow ^{236}\text{U}^* \begin{cases} \xrightarrow{84\%} \text{Fission} \\ \xrightarrow{16\%} {}^{236}\text{U} + \gamma \end{cases} \tag{2.49}$$

mass be u, and that of the second be $U = um/M$, with associated scattering angle θ. In the lab, the speed of m before the collision is

$$v_{\text{in}} = u + U \tag{2.53}$$

whereas after the collision it is given by v_{out} where

$$v_{\text{out}}^2 = v_{\text{out}\parallel}^2 + v_{\text{out}\perp}^2 = (u\cos\theta + U)^2 + (u\sin\theta)^2 = u^2 + 2Uu\cos\theta + U^2 \tag{2.54}$$

The fraction of energy lost by m in the lab frame is $v_{\text{out}}^2/v_{\text{in}}^2$. To find the average energy loss, we need to average over the scattering angle θ. Assuming the scattering occurs isotropically, we have that

$$\langle\cos\theta\rangle = -\int_{-1}^{1} d(\cos\theta)\ \cos\theta = 0 \tag{2.55}$$

That is, there is no preferred scattering angle. The average fractional energy lost by m is given by

$$\left\langle \frac{E_{\text{out}}}{E_{\text{in}}} \right\rangle = \frac{u^2 + U^2}{(u+U)^2} = \frac{m^2 + M^2}{(m+M)^2} \tag{2.56}$$

The above expression has a minimim at $m \equiv m_n M$; the ideal moderator contains nuclei that have mass comparable to that of the neutron. This is why H_2O, D_2O and graphite are relatively suitable for this use. They also have a relatively low cross-section for electron capture. In the ideal case that the moderator has $A = 1$, half of the initial neutron energy will be lost on average per collision. An initial 1 MeV neutron will then cool to 0.1 eV after approximately $\log_2(\text{MeV}/0.1\text{ eV}) \approx 23$ collisions.

Usually, lumps of fuel (rods) are embedded in a matrix of moderator material, rather than being completely mixed up with the moderator. This is because, since the mean free path of the fast electrons is less than the mean free path of the slow electrons ($\lambda_{\text{fast}} < \lambda_{\text{slow}}$), more of the fuel 'lump' can be seen by the thermal neutrons in this way. This would not be the case if the fuel and moderator were completely mixed up. It would also be very hard to extract power from the system if they were completely mixed together.

The ratio of the incoming to the outgoing partial widths is thus

$$\frac{\Gamma_i}{\Gamma_f} = \frac{\left|\langle \Delta^0 | \pi^- p \rangle\right|^2}{\left|\langle \Delta^0 | \pi^0 n \rangle\right|^2} = \frac{1/3}{2/3} = \frac{1}{2} \tag{3.13}$$

Conservation of isospin is a powerful tool that we will make further use of later on in this text.

3.1.2 Symmetries

Before delving further into this subatomic world, let us pause to consider possible symmetries of our system. A symmetry is a physical or mathematical feature of the system that is preserved or remains unchanged under some transformation. This transformation may be continuous (such as a rotation) or discrete (such as a reflection). In the case of continuous transformations, there give rise to cyclic coordinates, and a corresponding conserved quantity. For example, translational invariance implies momentum conservation, while rotational invariance implies angular momentum conservation. We shall examine some symmetries relevant to our study of subatomic particles.

Parity

A *parity inversion* $\mathcal{P}$ is a transformation involving the reversal of the sign of one spatial coordinate. In three dimensions, it describes a simultaneous switch of all of the coordinates (a point inversion):

$$\mathcal{P} : \mathbf{x} \mapsto -\mathbf{x} \tag{3.14}$$

We can classify vectors based on the way that they transform under a parity inversion:

- Polar - These reverse their sign under parity inversion (eigenvalue -1)
- Axial - These do not reverse their sign under parity inversion, often associated with cross-product two quantities (eigenvalue $+1$)

A *pseudoscalar* (such as a triple product) is a quantity that behaves like a scalar, except that it changes sign under parity inversion. Fermions and anti-fermions have opposite parity, and by conventions, fermions are assigned positive parity.

We know that for a single particle in a state of well-defined orbital angular momentum, applying a parity inversion to the state has the following effect:

$$\mathcal{P}\,|n, \ell, m_\ell\rangle = (-1)^\ell\,|n, \ell, m_\ell\rangle \tag{3.15}$$

The parity of the combined state of two particles must then be the product of the parities of the individual particles, and an extra factor from treating the two particles as a combined system. We can thus write that

$$\boxed{P_{ab} = P_a P_b (-1)^L} \tag{3.16}$$

where L is the total orbital angular momentum of the two particle system around the centre of mass, and P_{ab}, P_a, P_b are the parity eigenvalues of the combined state and the individual particles respectively.

Both the strong and electromagnetic interactions conserve parity. However, the weak interaction does not; not only does it not, but it violates parity conservation maximally. This can be tested by looking at the following beta decay process:

$$^{60}\mathrm{Co} \longrightarrow {}^{60}\,\mathrm{Ni} + e^- + \bar{\nu}_e \tag{3.17}$$

4.1 A Summary of The Standard Model

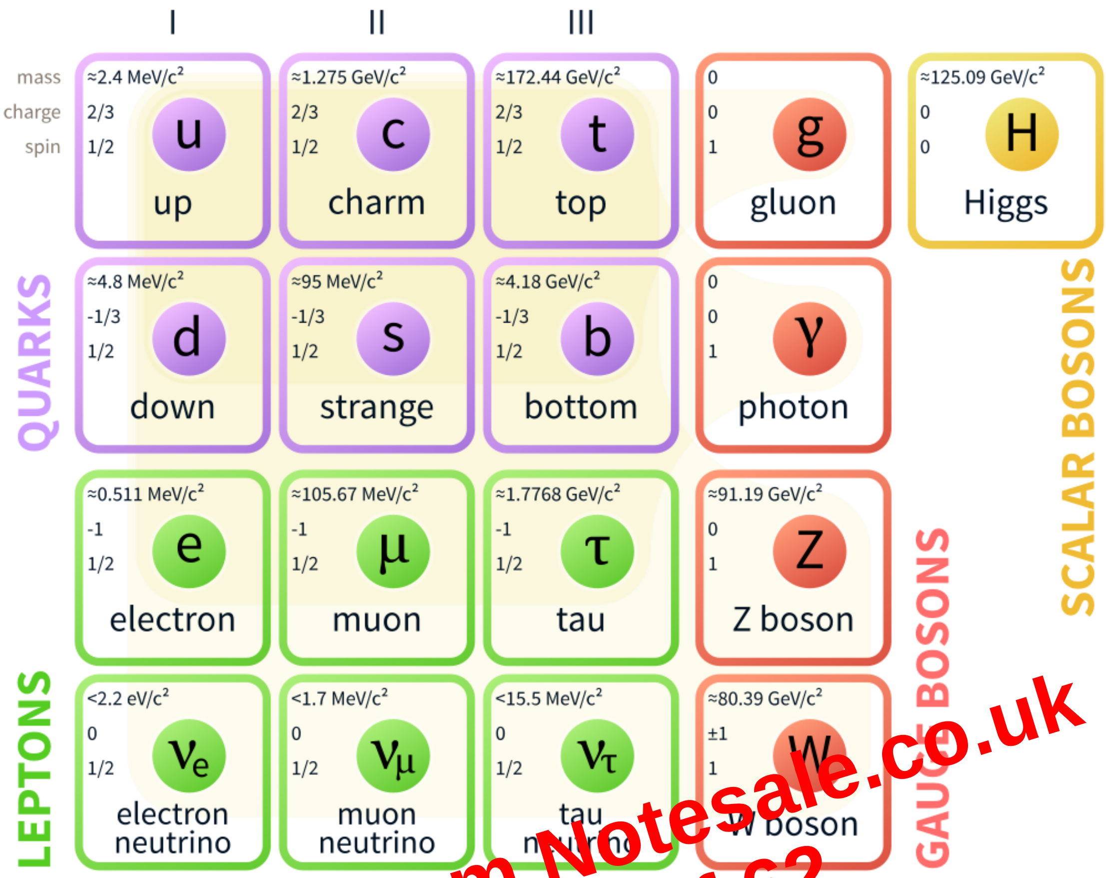

Figure 4.1: The canonical diagram for the Standard Model of Matter

The Standard Model, as it currently stands, consists of three main groups of particles. The spin-half quarks (u, d, c, s, t, b) make up all hadrons (mesons and baryons), matter particles that can undergo processes involving the strong interaction. The leptons (ℓ) are elementary spin-half particles that do not undergo processes involving the strong interaction. Mediating these interactions are the spin-one gauge bosons corresponding to the three (four) fundamental forces:

Force	Quantum	Mass[a]	Spin	Range	α	Strength[b]
Electromagnetic	Photon γ	0	1	∞	$1/137$	$\backsim 10^{-2}$
Strong	Gluon $(\times 8)$ g	0	1	$\backsim 10^{-15}$ m	$\backsim 1$	$\backsim 1$
Weak	$W^{\pm}$ Z^0	80.4 91.2	1 1	$\backsim 10^{-18}$ m	$1/29$	$\backsim 10^{-8}$
Gravity	Graviton?	0	2?	∞	-	$\backsim 10^{-41}$

The Higgs Boson is unique in that it interacts with the other particles with couplings proportional to their masses. It is an excitation of the vacuum Higgs field, responsible for giving mass to particles.

[a] in GeV
[b] Relative strength on the scale of the quarks

4.2.1 Deep Inelastic Scattering

Though we cannot isolate individual quarks, we are still able to perform scattering experiments at the quark level, known as *deep inelastic scattering.* For this, we need to use particles that are of sufficiently high energy ($\backsim$ GeV) such that their de-Broglie wavelength is comparable to the size of the proton (for example). This means that they will not scatter from the proton as a whole, instead undergoing scattering events with the individual quarks inside the proton.

These events pull the scattered quark away from the other quarks, which will create a hadron jet in the original direction of the outgoing quark if energies are sufficiently high. The other *spectator quarks* in the proton also form a coloured state, meaning that they must create hadrons in another jet event.

The interaction between the electron and the quark can either be mediated by the weak or electromagnetic forces. If the propagator is a $W^{\pm}$, then the collision is said to include a *weak charged current*, and the flavour of the scattered quark is changed. In the case of the Z^0 or γ, these are known as *weak neutral current* and *weak electromagnetic interaction* respectively, in which the quark flavour is unchanged.

At low values of the momentum transfer Q, the ratio of weak interactions to electromagnetic interactions is very small, where as at high values, it is of order unity. Consider the propagators:

$$G_\gamma = -\frac{1}{\mathsf{Q}^2}, \quad G_W = -\frac{1}{\mathsf{Q}^2 + m^2c^2} \tag{4.6}$$

For low values of Q, $m(W^{\pm}, Z^0)c^2 \gg \mathsf{Q}$ as the $W^{\pm}$ and Z^0 bosons are very massive, meaning that $G_\gamma \gg G_W$. However, as Q becomes very large, we can neglect the rest mass energies of these gauge bosons, such that the ratio of G_γ/G_W tends to unity.

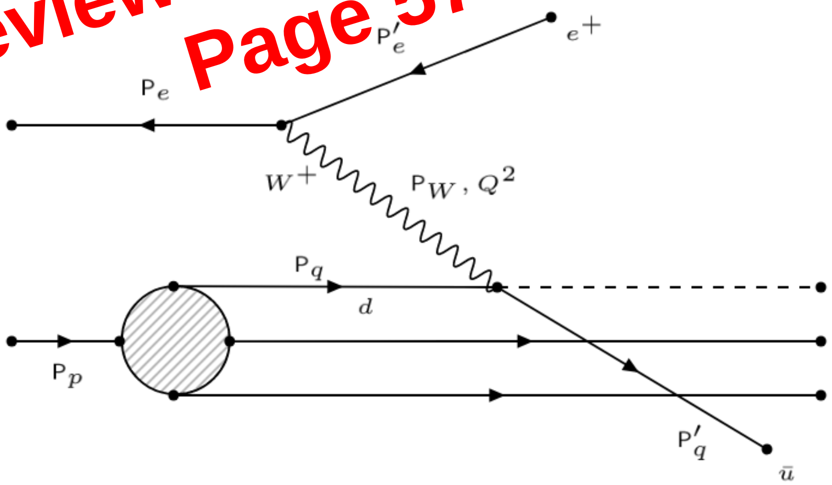

Figure 4.6: A positron-proton scattering event under weak charged current

Let us consider an example of a positron-proton scattering event in the deep-inelastic regime. We define the four-momenta

$$\mathsf{P}_e = (E_e/c, \mathbf{p}_e), \quad \mathsf{P}'_e = (E'_e/c, \mathbf{p}'_e), \quad \mathsf{P}_q = x\mathsf{P}_p \tag{4.7}$$

where we have defined the quark to have some fraction $0 < x < 1$ of the original proton momentum. Using invariants, we have that

$$-\,\mathsf{Q}^2 = \frac{(E_e - E_e')^2}{c^2} - (\mathbf{p}_e - \mathbf{p}_e')^2 \approx -\frac{2E_eE_e'}{c^2}(1-\cos\theta) \tag{4.8}$$

where the second expression follows from neglecting the rest mass energy of the positrons, as they are sufficiently relativistic in deep inelastic scattering events. θ is the angle through which the positron is scattered. Thus, the momentum transfer Q is given by

$$\mathsf{Q}^2 = \frac{4E_eE_e'}{c^2}\sin^2\left(\frac{\theta}{2}\right) \tag{4.9}$$

The centre of mass energy is given by

$$s = (\mathsf{P}_e + \mathsf{P}_q)^2c^2, \quad E_{CM} = \sqrt{s} \tag{4.10}$$

for the positron-quark system. Then,

$$s = (\mathsf{P}_e^2 + \mathsf{P}_q^2 + 2\mathsf{P}_e\cdot\mathsf{P}_q)c^2 = (m_e^2c^2 + x^2m_p^2c^2 + 2x\mathsf{P}_e\cdot\mathsf{P}_p)c^2 \tag{4.11}$$

We then assume that the particles are approximately massless at these relativistic energies, meaning that we can neglect the rest mass energies, and use

$$|\mathbf{p}_e| \approx \frac{E_e}{c}, \quad |\mathbf{p}_p| \approx \frac{E_p}{c}, \quad \mathbf{p}_e\cdot\mathbf{p}_p = -|\mathbf{p}_e||\mathbf{p}_p| \tag{4.12}$$

where we have assumed that the positron and proton collide head on. With these assumptions, we arrive at the final result of

$$\boxed{s = 4xE_eE_p \quad E_{CM} = \sqrt{s}} \tag{4.13}$$

Thus, the centre of mass energy available for the production of hadrons is proportional to the square root of the geometric mean of the energies of the positron and quark.

Detectors

How do we detect the particles produced in the jets associated with deep inelastic scattering? The momentum of the antielectron can be determined using the inner part of the detector (the *tracker*) that tracks the trajectory of charged particles, and measures their momenta as they bend in an externally applied field. The direction of curvature of the track (relative to the magnetic field) allows the sign of the charge to be deduced, while the radius of curvature R permits the calculation of the perpendicular component of the momentum from

$$|\mathbf{p}_\perp| = qBR \tag{4.14}$$

The *calorimeter* lies outside the tracker, and is designed to stop particles and convert their energies into an electrical signal. This allows the other components of the momentum to be determined.

Neutrinos (where produced) cannot be directly detected in such scattering experiments as they are so weakly interacting. However, we can determine the perpendicular component of their momentum from momentum conservation in the perpendicular plane.

4.3 The Weak Interaction

The weak interaction is mediated by three spin-one bosons: the charged $W^{\pm}$ (antiparticles of one another), and the neutral Z^0 (its own antiparticle). The weak force is very short ranged as a result of their large masses:

$$m_W = 80.3\,\mathrm{GeV}/c^2, \quad m_Z = 91.2\,\mathrm{GeV}/c^2 \tag{4.15}$$

from which we approximate their range $\hbar c/mc^2$ is about 2×10^{-3} fm, which is about 1000 times smaller that the size of a proton. As stated previously, the coupling of leptons to the W boson is universal, with vertex factor g_W. This means that decays of $W^{\pm}$ into any of the three lepton generations is equally likely (as the density of states is roughly the same in these highly relativistic decays).

4.3.1 Quark Mixing

The weak interaction is the only type of interaction that can change quark flavour, which only occurs in interactions mediated by the $W^{\pm}$ bosons.

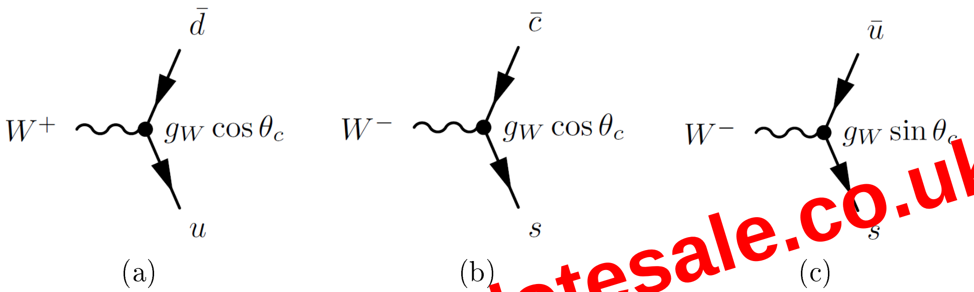

Figure 4.7: The coupling of the $W^{\pm}$ bosons to quarks (a) $u\bar{d}$ (b) $\bar{c}s$ (c) $\bar{u}s$ (suppressed)

A vertex involving a $W^{\pm}$ and two quarks must include an upper $Q = 2/3$ and lower $Q = -1/3$ quark, in order to conserve charge. The vertices that include two quarks of the same generation dominate, while those involving transitions between those of different generations are suppressed. To see why this is, let us consider each of the 'lower' quarks to have their our weak interaction eigenstates $|d'\rangle$, $|s'\rangle$ and $|b'\rangle$, each of which is a superposition of the mass eigenstates $|d\rangle$, $|s\rangle$ and $|b\rangle$ (the observable eigenstates). The $|u\rangle$, $|c\rangle$ and $|t\rangle$ couple to the $|d'\rangle$, $|s'\rangle$ and $|b'\rangle$ states respectively.

Let us consider the simple case of just the down and strange quarks. In the interaction basis, the couplings of the $W^{\pm}$ are diagonal, so the (W, u, d') and the (W, c, s') couplings take the same value g_W, where as the couplings (W, u, s') and (W, c, d') are each zero. The flavour eigenstates $\{|d'\rangle, |s'\rangle\}$ in the primed basis must differ from the mass eigenstates $\{|d\rangle, |s\rangle\}$ in the unprimed basis, as we do indeed observe cross-generational quark mixing. This is done by a rotation:

$$\boxed{\begin{pmatrix} |d'\rangle \\ |s'\rangle \end{pmatrix} = V_c \begin{pmatrix} |d\rangle \\ |s\rangle \end{pmatrix}, \quad V_c = \begin{pmatrix} \cos\theta_c & -\sin\theta_c \\ \sin\theta_c & \cos\theta_c \end{pmatrix}} \tag{4.16}$$

V_c is the 2×2 *Cabibbo rotation matrix* (or *CKM* matrix), while $\theta_c \approx 13°$ is the *Cabibbo rotation angle*. As this rotation angle is small, this means that interactions that change generation are heavily suppressed, while interactions that do not change generation are not. If we include the full basis, this becomes a 3×3 matrix, with couplings between generations I and III being suppressed.