Contents

- Slow Motion - Objects move slowly compared to the speed of light, such that $\mathbf{v} \ll c$, or alternatively that $cdt \gg dx^i$, were the index i refers to the spatial components of the metric (we shall adopt i and j for this purpose throughout the remainder of this text). This means that in the Newtonian limit, $\tau \approx t$
- Slowly Varying Gravitational Fields - This means that any time derivatives of the metric can be ignored

Applying this to the geodesic equation, the only component of the metric that we are interested in is g_{00}, such that the only affine connection that is non-vanishing is Γ^ρ_{00}:

$$\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho_{00}\left(\frac{dx^0}{d\tau}\right)^2 \approx 0 \tag{1.38}$$

Using the fact that the gravitational field is roughly stationary, and (1.37), we can re-write the connection as

$$\Gamma^\rho_{00} = -\frac{1}{2}g^{\rho\mu}\frac{\partial g_{00}}{\partial x^\mu} \approx -\frac{1}{2}\eta^{\rho\mu}\frac{\partial h_{00}}{\partial x^\mu} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^\rho} \tag{1.39}$$

where we have made use of the fact that only spatial components of $\eta^{\rho\mu}$ remain, which are equal to unity. Then, the geodesic equation becomes

$$\frac{d^2x^\rho}{d\tau^2} = \frac{1}{2}\left(\frac{dx^0}{d\tau}\right)^2\frac{\partial h_{00}}{\partial x^\rho} \quad \longrightarrow \quad \frac{d^2\mathbf{x}}{d\tau^2} = \frac{1}{2}\left(\frac{dx^0}{d\tau}\right)^2 \nabla h_{00} \tag{1.40}$$

Comparing this with the Newtonian result $d^2\mathbf{x}/dt^2 = -\nabla\Phi$, we see that in the Newtonian limit

$$\boxed{g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right), \quad h_{00} = -\frac{2\Phi}{c^2}} \tag{1.41}$$

Note that for metrics containing $\Phi(r) = -GM/r$, this expression can be found by expanding them for small Φ.

1.2.3 Time Dilation

We can write any metric explicitly in the form

$$-c^2d\tau^2 = g_{00}\,c^2dt^2 + g_{ij}dx^idx^j \tag{1.42}$$

where again the indices i and j are used to refer to the spatial components of the metric. Then, it is clear that the proper time, and the local inertial time are related by

$$\boxed{\frac{dt}{d\tau} = \left(-g_{00} - \frac{1}{c^2}g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right)^{-1/2}} \tag{1.43}$$

This is the analogue of time dilation in General Relativity. In particular, if the clock measuring t is at rest in the corresponding inertial frame, this becomes

$$\frac{dt}{d\tau} = (-g_{00})^{-1/2} \tag{1.44}$$

Now, consider two observes A and B that are tangent to $K^0 = (1,0,0,0)$, such that they are both at rest in some frame. For each observer, it is clear that

$$dt = d\tau_A(-g_{00}(A))^{-1/2}, \quad dt = d\tau_B(-g_{00}(B))^{-1/2} \tag{1.45}$$

This means that the observed frequencies are related by

$$\frac{\nu_B}{\nu_A} = \frac{d\tau_A}{d\tau_B} = \sqrt{\frac{g_{00}(A)}{g_{00}(B)}} \approx 1 - \frac{\Phi_B - \Phi_A}{c^2} \tag{1.46}$$

where we have assumed that we are in the Newtonian limit; we have thus obtained the previous gravitational redshift result directly from the metric.

special-relativistic equations that hold in the absence of gravitation, replace $\eta_{\mu\nu}$ with $g_{\mu\nu}$, and promote all derivatives to covariant derivatives. The resulting equations will be generally covariant and true in the absence of gravitation, and therefore (according to the principle of General Covariance), they will be true in the presence of gravitational fields, provided that we always work on a spacetime scale sufficiently small compared with the scale of variation of the gravitational field.

More on the Covariant Derivative

In a similar way to partial derivatives, we can also define the *covariant divergence* as

$$\nabla_\mu \mathsf{U}^\mu = \partial_\mu \mathsf{U}^\mu + \Gamma^\mu_{\mu\nu}\mathsf{U}^\nu \tag{1.55}$$

Recalling (1.25), we can write that

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2g_{\mu\rho}}\partial_\nu g_{\rho\mu} = \frac{1}{2}\partial_\nu \log(-g) \tag{1.56}$$

where g is the determinant of $g_{\mu\nu}$, which for diagonal metrics is simply the product of entries. Though the above equation seems specific to diagonal metrics, it is in fact a general result. Then, our covariant divergence becomes

$$\boxed{\nabla_\mu \mathsf{U}^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,\mathsf{U}^\mu\right)} \tag{1.57}$$

We can write a similar expression for the covariant divergence of a tensor:

$$\nabla_\mu \mathbb{T}^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,\mathbb{T}^{\mu\nu}\right) + \Gamma^\nu_{\mu\rho}\mathbb{T}^{\mu\rho} \tag{1.58}$$

For an antisymmetric tensor, the last term disappears, as the connection is symmetric in its lower indices. We shall make use of this expression in due course.

Parallel Transport

Given that $\mathsf{U}^\mu = dx^\mu/d\tau$ is a tangent vector to our spacetime coordinates, we define the absolute derivative of another vector V^ρ along a path $x^\mu(\tau)$ as

$$\frac{D\mathsf{V}^\rho}{D\tau} = \mathsf{U}^\mu\nabla_\mu\mathsf{V}^\rho \equiv \nabla_{\mathsf{U}}\mathsf{V}^\rho, \quad \frac{D}{D\tau} = \mathsf{U}^\mu\nabla_\mu = \nabla_{\mathsf{U}} \tag{1.59}$$

We say that the vector V^μ undergoes *parallel transport* when moving along this curve in the case that $D/D\tau = 0$. For example, a vector may always point along $\mathbf{e}_y$ as we move it in the xy plane, but its corresponding r and θ components will constantly have to change to keep this true. This is embodied in the parallel transport condition. Now, if we apply this to the tangent vector U^μ, then we have that

$$\mathsf{U}^\mu\nabla_\mu\mathsf{U}^\rho = \mathsf{U}^\mu\left(\partial_\mu\mathsf{U}^\rho + \Gamma^\rho_{\mu\nu}\mathsf{U}^\nu\right) = \frac{D\mathsf{U}^\rho}{D\tau} = 0 \tag{1.60}$$

by the definition of U^μ. However, we can recognise the second expression as the geodesic equation; this means that an alternative way of writing it is

$$\boxed{\mathsf{U}^\mu\nabla_\mu\mathsf{U}^\rho = 0, \quad \mathsf{U}^\rho = \frac{dx^\rho}{d\tau}} \tag{1.61}$$

1.4 Einstein's Field Equations

We spent the last section investigating how to formalise the way in which spacetime curves through the Riemann curvature tensor. However, we have not yet outlined how this curvature relates to the presence of mass. This relationship is embodied in the *Einstein field equations*, which we will now seek to derive using this vast theoretical apparatus that we have constructed.

As previously suggested, the source term for gravitation in General Relativity is the stress energy tensor $\mathbb{T}^{\mu\nu}$. Now, we must relate curvature to this in a way that means that the conservation condition $\mathbb{T}^{\mu\nu}_{;\mu} = 0$ is still satisfied. We already have a neat form for this given by (1.98). We thus propose the solution of the form

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = c_1\mathbb{T}^{\mu\nu} + c_2 g^{\mu\nu} \tag{1.110}$$

for constants c_1 and c_2. We can determine these constants by looking at the Newtonian limit of this equation. Once again, the dominant terms are $\mathbb{T}^{00} = \rho c^2$ and g^{00}, and we can use the weak field condition (1.37) to write that

$$\Gamma^\rho_{\sigma\nu} = \frac{1}{2}\eta^{\rho\mu}(\partial_\sigma h_{\mu\nu} + \partial_\nu h_{\mu\sigma} - \partial_\mu h_{\sigma\nu}) \tag{1.111}$$

$$R^\rho{}_{\mu\sigma\nu} = \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu\partial_\sigma h_{\nu\lambda} + \partial_\nu\partial_\lambda h_{\rho\mu} - \partial_\lambda\partial_\sigma h_{\mu\nu} - \partial_\mu\partial_\nu h_{\rho\lambda}) \tag{1.112}$$

$$R_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu\partial_\rho h_{\nu\lambda} + \partial_\nu\partial_\lambda h_{\rho\mu} - \partial_\lambda\partial_\rho h_{\mu\nu} - \partial_\mu\partial_\nu h_{\rho\lambda}) \tag{1.113}$$

To find c_1, we focus on the R^{00} component of the field equation. Assuming the fields are slowly varying, such that all time derivatives of the metric vanish, meaning that we are left with

$$R_{00} = -\frac{1}{2}\partial_\lambda\partial_\rho h_{00} = -\frac{1}{2}\nabla^2 h_{00} = \frac{1}{c^2}\nabla^2\Phi \tag{1.114}$$

where we have made use of (1.41). For the Ricci scalar, we let $c_2 = 0$, and assume that $\mathbb{T}_{ij} \approx 0$ (as once again the timelike component is dominant), such that

$$R_{ij} = \frac{1}{2}g_{ij}R = \frac{1}{2}\delta_{ij}R \tag{1.115}$$

Then, the Ricci scalar is

$$R = g^{\mu\nu}R_{\mu\nu} \approx \eta^{\mu\nu}R_{\mu\nu} = -R_{00} + R_{ii} = -R_{00} + \frac{3}{2}R \quad\longrightarrow\quad R = 2R_{00} \tag{1.116}$$

This means that the timelike component of the field equation becomes

$$2R_{00} = c_1\mathbb{T}_{00} \quad\longrightarrow\quad \nabla^2\Phi = c_1\frac{\rho c^4}{2} \tag{1.117}$$

Comparison with $\nabla^2\Phi = 4\pi G\rho$ means that our constant $c_1 = 8\pi G/c^4$. It is convention to define $c_2 = -\Lambda$, known as the *cosmological constant*. Finally, we have arrived at the Einstein field equations:

$$\boxed{G^{\mu\nu} = \frac{8\pi G}{c^4}\mathbb{T}^{\mu\nu} - \Lambda g^{\mu\nu}, \quad G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R} \tag{1.118}$$

where we have introduced the field tensor $G^{\mu\nu}$. The field equations completely encapsulate the interaction between matter and spacetime curvature, and vice-versa. In essence, it describes how "space tells matter how to move, and matter tells space how to move", in a beautiful summary of the non-linear theory that is General Relativity

2.1 The Schwarzchild Solution

Let us re-write the field equation in terms of the source term $\mathbb{T}^{\mu\nu}$:

$$R^{\mu\nu} = \frac{8\pi G}{c^4}\mathbb{S}^{\mu\nu} - \Lambda g^{\mu\nu}, \quad \mathbb{S}^{\mu\nu} = \left(\mathbb{T}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\right) \tag{2.1}$$

where $T = \mathbb{T}^{\mu}_{\mu}$. We shall assume that we are in a vacuum far from source terms ($\mathbb{S}^{\mu\nu} = 0$), and shall ignore the cosmological constant factor Λ (as this only becomes relevant when working on cosmological scales, as in chapter 4). This means that our curvature is described by the vacuum field equation:

$$R_{\mu\nu} = 0 \tag{2.2}$$

Suppose that we want to find a spherically symmetric metric that satisfies the above equation vacuum equation, and is asymptotically flat as $r \to \infty$. We thus propose a diagonal metric of the form

$$-c^2 d\tau^2 = -B(r,t)c^2dt^2 + A(r,t)dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{2.3}$$

where A and B are arbitrary functions of r and t. Now, in order to find functional forms for A and B, we must impose the constraints given by the components of (2.2), for which we need to find the Ricci tensor. Let us begin by computing all the non-zero components of the affine connection using (1.25) and (1.26):

$$\begin{aligned}
\Gamma^r_{00} &= -\frac{1}{2}g^{rr}\partial_r g_{00} = \frac{B'}{2A} \\
\Gamma^r_{rr} &= -\frac{1}{2}g^{rr}\partial_r g_{rr} = \frac{A'}{2A} \\
\Gamma^r_{\theta\theta} &= -\frac{1}{2}g^{rr}\partial_r g_{\theta\theta} = -\frac{r}{A} \\
\Gamma^r_{\phi\phi} &= -\frac{1}{2}g^{rr}\partial_r g_{\phi\phi} = -\frac{r\sin^2\theta}{A} \\
\Gamma^r_{r0} &= \frac{1}{2}g^{rr}\partial_0 g_{rr} = \frac{\dot{A}}{2A}
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\Gamma^\theta_{\theta r} &= \frac{1}{2}g^{\theta\theta}\partial_r g_{\theta\theta} = \frac{1}{r} \\
\Gamma^\theta_{\phi\phi} &= -\frac{1}{2}g^{\theta\theta}\partial_r g_{\phi\phi} = \sin\theta\cos\theta \\
\Gamma^\phi_{\phi r} &= \frac{1}{2}g^{\phi\phi}\partial_r g_{\phi\phi} = \frac{1}{r} \\
\Gamma^\phi_{\phi\theta} &= -\frac{1}{2}g^{\phi\phi}\partial_\theta g_{\phi\phi} = \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

$$\Gamma^0_{00} = \frac{1}{2}g^{00}\partial_0 g_{00} = \frac{\dot{B}}{2B} \tag{2.5}$$

$$\Gamma^0_{0r} = \frac{1}{2}g^{00}\partial_r g_{00} = -\frac{B'}{2B}$$

$$\Gamma^0_{rr} = \frac{1}{2}g^{00}\partial_0 g_{rr} = \frac{\dot{A}}{2B} \tag{2.6}$$

Note that we have used a dash to refer to a derivative with respect to r, while a dot indicates a derivative with respect to t. Note that the only connections where a time derivative enters are (2.4), (2.5) and (2.6).

Using (1.25) and (1.26), we can write the Ricci tensor (1.89) in the useful form

$$\boxed{R_{\mu\nu} = \frac{1}{2}\partial_\mu\partial_\nu \log(-g) - \partial_\rho \Gamma^\rho_{\mu\nu} + \Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \frac{1}{2}\Gamma^\lambda_{\mu\nu}\partial_\lambda \log(-g)} \tag{2.7}$$

where as usual $g = -ABr^4 \sin^2\theta$ is the determinant of $g_{\mu\nu}$. Plugging the affine connections into this (the algebra is left as an exercise for the reader; had to get that phrase in here somewhere), we find that

$$R_{00} = -\frac{B''}{2A} + \frac{B'}{4A}\left[\frac{B'}{B} + \frac{A'}{A}\right] - \frac{B'}{rA} + \frac{1}{2}\left[\frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2} + \frac{\dot{A}\dot{B}}{2AB} + \frac{\dot{A}B'}{2AB} + \frac{\dot{A}A'}{2A^2} + \frac{\dot{B}^2}{2AB}\right] \tag{2.8}$$

$$R_{rr} = \frac{B''}{2B} - \frac{A'B'}{4AB} - \frac{B'^2}{4B^2} - \frac{A'}{rA} - \frac{1}{2}\left[\frac{\dot{A}}{2B}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) - \frac{\dot{A}B'}{AB} + \frac{\dot{B}^2}{2B^2}\right] \tag{2.9}$$

$$R_{0r} = -\frac{\dot{A}}{2A} \tag{2.10}$$

$$R_{0\theta} = R_{0\phi} = R_{r\phi} = R_{r\theta} = 0 \tag{2.11}$$

Note that we have not evaluated $R_{\theta\theta}$ and $R_{\phi\phi}$ as these do not actually provide constraints on A or B. This could have been guessed at by the form of the metric (2.3). Now, all of these terms have to be identically zero according to (2.2). This immediately implies that $\dot{A} = 0$, such that R_{00} and R_{rr} become

$$R_{00} = -\frac{B''}{2A} + \frac{B'}{4A}\left[\frac{B'}{B} + \frac{A'}{A}\right] - \frac{B'}{rA} + \frac{\dot{B}^2}{4AB} \tag{2.12}$$

$$R_{rr} = \frac{B''}{2B} - \frac{A'B'}{4AB} - \frac{B'^2}{4B^2} - \frac{A'}{rA} - \frac{\dot{B}^2}{4B^2} \tag{2.13}$$

Now, consider the following combination

$$\frac{R_{00}}{B} + \frac{R_{rr}}{A} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0 \tag{2.14}$$

The remaining term involving time derivatives of B has vanished. This equation implies that

$$AB = \text{constant} = 1 \tag{2.15}$$

where the constant has been fixed by arguing that both A and B should return to their Minkowski values as $r \to \infty$. As $B = A^{-1}$, this further means that B must also be independent of time. This is statement of *Birkhoff's theorem*: that any spherically symmetric solution of the vacuum field equation (2.2) must be static, and asymptotically flat. This is interesting; even when we introduced a possible time dependence into our metric, we were forced by the nature of the equations describing spacetime to abandon it!

The condition $R_{00} = 0$ now gives us that

$$B'' + \frac{2B'}{r} = 0 \tag{2.16}$$

meaning that B is a linear combination of a constant and another constant times $1/r$. However, we have the additional constraint that B must approach the Newtonian limit (1.41) for $r \to \infty$. This leads us to conclude that

$$B(r) = 1 - \frac{2GM}{rc^2}, \quad A(r) = \left(1 - \frac{2GM}{rc^2}\right)^{-1} \tag{2.17}$$

2.2 Orbits around a Point Mass

Now that we have an explicit metric to work with, we can begin investigating how objects move within said metric. In most cases, we will be considering the trajectories of light objects moving in the gravitational field of much larger objects, such as stars or black holes.

2.2.1 Classical Orbits

Before tackling the problem of orbits in the Schwarzchild metric, let us refresh our knowledge of classical orbits. Recall the classical orbit equation

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{2r^2} - \frac{GM}{r} = \mathcal{E}_0 \tag{2.24}$$

which we can derive from energy conservation considerations. $J = r^2 d\phi/dt$ is the (constant) angular momentum per unit mass of the orbiting body, while $\mathcal{E}_0$ is some constant energy corresponding to the initial kinetic energy per unit mass of the orbiting body. We can then write that

$$\frac{dr}{dt} = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{J}{r^2}\frac{dr}{d\phi} = \pm\left[2\mathcal{E}_0 + \frac{2GM}{r} - \frac{J^2}{r^2}\right]^{1/2} \tag{2.25}$$

We can then integrate both sides of this equation:

$$\pm\,\phi = \int dr\,\frac{J}{r^2\left[2\mathcal{E}_0 + \frac{2GM}{r} - \frac{J^2}{r^2}\right]^{1/2}} = \int du\,\frac{1}{\left[\frac{2\mathcal{E}_0}{J^2} + \frac{2GM}{J^2}u - u^2\right]^{1/2}} \tag{2.26}$$

The second expression follows from making the usual substitution $u = r^{-1}$. With some re-arrangement, this becomes a recognisable integral:

$$\pm\,\phi = \int du\,\frac{1}{\left[\frac{2\mathcal{E}_0}{J^2} + \frac{G^2M^2}{J^4} - \left(u - \frac{GM}{J^2}\right)^2\right]^{1/2}} = \cos^{-1}\left[\frac{u - \frac{GM}{J^2}}{\left(\frac{2\mathcal{E}_0}{J^2} + \frac{G^2M^2}{J^4}\right)^{1/2}}\right] \tag{2.27}$$

meaning that we can write this as

$$u - \frac{GM}{J^2} = \frac{GM}{J^2}\left(1 + \frac{2\mathcal{E}_0 J^2}{G^2M^2}\right)^{1/2}\cos\phi \tag{2.28}$$

Defining the *elliptical eccentricity* e, we can recover the solution for r in terms of constants of the motion, and the phase angle $\phi = \Omega t$.

$$\boxed{r = \frac{J^2}{GM(1 + e\cos\phi)}, \quad e = \left(1 + \frac{2\mathcal{E}_0 J^2}{G^2M^2}\right)^{1/2}} \tag{2.29}$$

Note that we often define the *latus-rectum* ℓ such that

$$r = \frac{\ell}{1 + e\cos\phi}, \quad r_+ = \frac{\ell}{1+e}, \quad r_- = \frac{\ell}{1-e} \tag{2.30}$$

Then, supposing that the *semi-major* axis of the ellipse is a, it follows that

$$a = \frac{1}{2}(r_+ + r_-) \quad \longrightarrow \quad \ell = a(1-e^2), \quad J = [GMa(1-e^2)]^{1/2} \tag{2.31}$$

This means that we can write the rate of change of ϕ in terms of constants of the motion, and r:

$$\boxed{\Omega = \frac{d\phi}{dt} = \frac{J}{r^2} = \frac{[GMa(1-e^2)]^{1/2}}{r^2}} \tag{2.32}$$

We shall move back to the Schwarzchild metric to tackle this problem. For simplicity, assume that the orbit of the gas is circular. A distant observer with coordinate time t will observe the rotating gas moving with a velocity

$$v = r\frac{d\phi}{dt} = c\sqrt{\frac{r_s}{2r}} \tag{2.69}$$

where we have made use of (2.45). Then, the local Doppler frequency shift due to this motion is given by

$$\frac{\omega_r}{\omega_e} = \sqrt{\frac{c \pm v}{c \mp v}} \tag{2.70}$$

where ω_r is the frequency observed at some radius r near the black hole, and ω_e is the emitted frequency of the radiation in the rest frame of the source. The positive and negative signs correspond to matter moving towards and away from the observer along the line of sight. For light, the (normalised) tangent to the null geodesic is given by

$$\mathsf{U}^\mu = \left(1 - \frac{r_s}{r}\right)^{-1/2}(1, 0, 0, 0) \tag{2.71}$$

as $K^0 = \partial/\partial t$ is Killing in Schwarzchild. Let P^μ be the four-momentum of the emitted radiation. Then, we can write the invariant quantity

$$\mathsf{P}_\mu \mathsf{U}^\mu = \left(1 - \frac{r_s}{r}\right)^{-1/2}\mathsf{P}^0 \tag{2.72}$$

From this, it clearly follows that

$$\frac{\omega_\infty}{\omega_r} = \left(1 - \frac{r_s}{r}\right)^{1/2} \tag{2.73}$$

where ω_∞ is the frequency observed at $r \to \infty$. Thus we find an equation for the relationship between the frequency emitted in the rest frame of the gas located at some radius r, and that observed by a distant observer far from the black hole:

$$\boxed{\frac{\omega_\infty}{\omega_e} = \left[\left(1 - \frac{r_s}{r}\right)\left(\frac{1 \pm \sqrt{r_s/2r}}{1 \mp \sqrt{r_s/2r}}\right)\right]^{1/2}} \tag{2.74}$$

where again the positive and negative signs correspond to matter moving towards and away from the observer along the line of sight. It is then easy to show that the line broadening is given by

$$\frac{\Delta\omega}{\omega_e} = 2\left(1 - \frac{r_s}{r}\right)^{1/2}\left(\frac{2r}{r_s} - 1\right)^{-1/2} \tag{2.75}$$

Suppose that the gas is located at the minimum stable circular orbit, so $r = 3r_s$, and thus $\Delta\omega/\omega_e = (8/15)^{1/2}$. The broadening is thus comparable to the emitted frequency close to the black hole.

It is clear that this reduces to (3.71) in the limit that $e \to 0$. The advantage of this result is that it is applicable to all types of orbits, even very eccentric ones. For example, we can consider a single parabolic encounter between two massive bodies. Such a scattering event corresponds to $e \to 1$ in such a way that $\ell = a(1-e^2) = a(1-e)(1+e)$ remains finite. The distance of closest approach is given by $b = a(1-e)$, while the period of the orbit is given by

$$T = 2\pi\sqrt{\frac{a^2}{GM}} \tag{3.82}$$

The total gravitational energy emitted in the encounter is given by

$$\Delta E_{GM} = \lim_{e\to 1} T\,\langle L_{GW}\rangle \tag{3.83}$$

which is give explicitly by

$$\Delta E_{GW} = \frac{84\sqrt{2}\pi}{24}\frac{G^{7/2}m_1^2 m_2^2(m_1+m_2)}{c^5 b^{7/2}} \tag{3.84}$$

Limits on Gravitational Radiation

According to Hawking (1971), when two black holes of masses m_1 and m_2 collide to form a single large black hole of mass M, the total area of the event horizon must increase. This is due to causality arguments. We know from our work with null radial geodesics in section 2.3.2 that points inside Schwarzchild radii of the original two black holes cannot lie outside of the Schwarzchild radius of the final black hole, as otherwise this would correspond to the propagation of something outwards across the event horizon. One can show that this places the constraint on the merger that the total area of the event horizon must increase.

We know that the invariant volume is given by

$$d^4V = \sqrt{-g}\,d^4x = \sqrt{-g}\,dt\,dr\,d\theta\,d\phi \tag{3.85}$$

Let us adopt the Schwarzchild metric (2.18), such that $g = -c^2r^4\sin^2\theta$. Then, the surface area element in our space is given by

$$dS = \bar{\partial}_t\bar{\partial}_r\sqrt{-g}\,d^4x \tag{3.86}$$

where

$$\bar{\partial}_t = \frac{1}{c(1-r_s/r)^{1/2}}\partial_t, \quad \bar{\partial}_r = (1-r_s/r)^{1/2}\partial_r \tag{3.87}$$

are the normalised tangent vectors in Schwarzchild geometry. This evaluates to

$$dS = r^2\sin\theta d\theta d\phi \tag{3.88}$$

We could have immediately deduced this from the fact that the metric is spherically symmetric, but we have shown this method here as it is easily generalised. From this, it is clear that the area of the event horizon is given by

$$A = 4\pi r_s^2 \tag{3.89}$$

We must have that the final area is greater than the initial area, giving us the following constraint on the masses

$$M \geq \left(m_1^2 + m_2^2\right)^{1/2} \tag{3.90}$$

It follows that the upper limit on the energy that can be emitted as gravitational radiation in such a black hole merger is

$$\Delta E_{GW}/c^2 \leq m_1 + m_2 - \left(m_1^2+m_2^2\right)^{1/2} \tag{3.91}$$

4.2 The Friedmann Equations

Now that we have a metric to describe the universe, we can in principle solve the field equations (1.118), which we shall now do. Unlike with the Schwarzchild solution, we shall retain the cosmological constant Λ, as this is important on cosmological scales.

4.2.1 The Friedmann Solution

Recall that the components of the FRW metric are

$$g_{00} = -1, \quad g_{rr} = \frac{a^2}{1-kr^2}, \quad g_{\theta\theta} = a^2r^2, \quad g_{\phi\phi} = a^2r^2\sin^2\theta \tag{4.27}$$

Adopting the notation $\dot{a} = da/dt$, the non-zero components of the affine connection are

$$\begin{aligned}
\Gamma^0_{rr} &= -\frac{1}{2}g^{00}\partial_0 g_{rr} = \frac{a\dot{a}}{1-kr^2} \\
\Gamma^0_{\theta\theta} &= -\frac{1}{2}g^{00}\partial_0 g_{\theta\theta} = a\dot{a}r^2 \\
\Gamma^0_{\phi\phi} &= -\frac{1}{2}g^{00}\partial_0 g_{\phi\phi} = a\dot{a}r^2\sin^2\theta
\end{aligned}$$

$$\begin{aligned}
\Gamma^r_{\theta\theta} &= -\frac{1}{2}g^{rr}\partial_r g_{\theta\theta} = -r(1-kr^2) \\
\Gamma^r_{\phi\phi} &= -\frac{1}{2}g^{rr}\partial_r g_{\phi\phi} = -r(1-kr^2)\sin^2\theta
\end{aligned}$$

$$\begin{aligned}
\Gamma^\theta_{\phi\phi} &= -\frac{1}{2}g^{\theta\theta}\partial_\theta g_{\phi\phi} = -\sin\theta\cos\theta \\
\Gamma^\phi_{\theta\phi} &= \frac{1}{2}g^{\phi\phi}\partial_\theta g_{\phi\phi} = \cot\theta
\end{aligned}$$

$$\begin{aligned}
\Gamma^r_{0r} &= \Gamma^\theta_{0\theta} = \Gamma^\phi_{0\phi} = \frac{\dot{a}}{a} \\
\Gamma^\theta_{r\theta} &= \Gamma^\phi_{r\phi} = \frac{1}{r}
\end{aligned}$$

Then, using (2.7), the components of the Ricci tensor are

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad R_{ij} = (\ddot{a}a + 2\dot{a}^2 + 2k)g_{ij} \tag{4.28}$$

with the Ricci scalar being

$$R = \frac{6}{a^2}(\ddot{a}a + \dot{a}^2 + k) \tag{4.29}$$

We note that the curvature along t = constant slices in the space is $R = 6k/a^2$, which is consistent with either flat, positively or negatively curved space for $k = 0$, $k = 1$, and $k = -1$ respectively.

Fluid Conservation

The universe is evidently not empty (or else how would this author be sitting here writing this?), so we are not interested in vacuum solutions to the fields equations. The assumptions

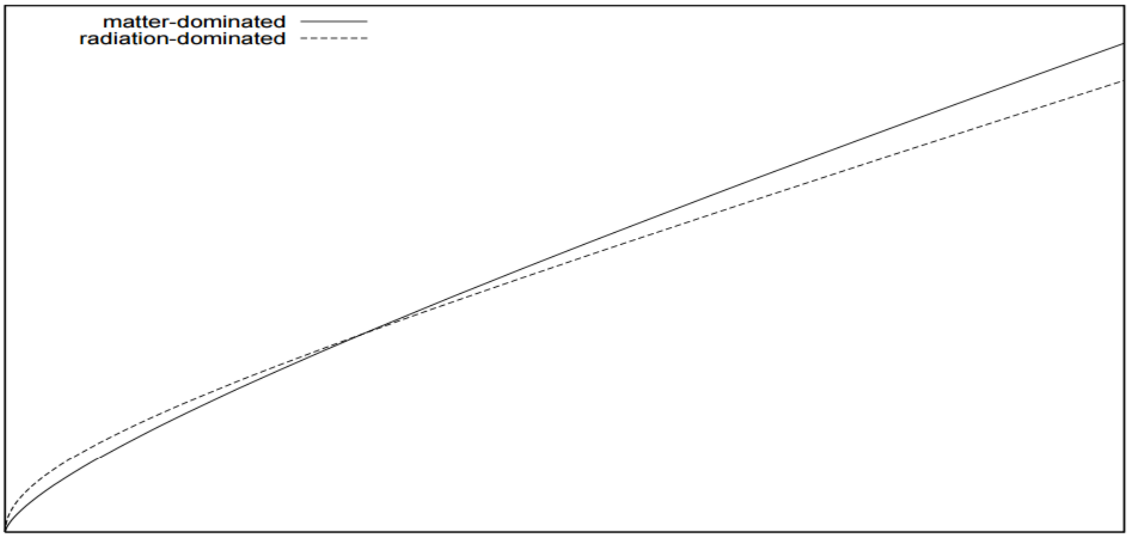

Figure 4.2: A plot of the scale factor (vertical axis) against time (horizontal axis) for the matter dominated and radiation dominated cases of an open universe

Closed Universes

For closed universes, $k = 1, \Omega_k < 0$. We again consider two special cases:

- Matter dominated (Ω_m):

$$\int dt = \frac{1}{H_0}\int da\, \frac{1}{a\left[\Omega_m a^{-3} + \Omega_k a^{-2}\right]^{1/2}} = \int da\, \frac{1}{\left[|I_{mk}|\, a^{-1} - 1\right]^{1/2}} \tag{4.80}$$

where I_{mk} is defined as before. We have included the absolute value, as by definition Ω_k is negative for closed universes. Using conformal time:

$$\int d\eta = \int da\, \frac{1}{\left[|I_{mk}|\, a - a^2\right]^{1/2}} \quad \longrightarrow \quad \eta = \sin^{-1}\left(\frac{2a - |I_{mk}|}{|I_{mk}|}\right) + \frac{\pi}{2} \tag{4.81}$$

such that

$$\frac{2a - |I_{mk}|}{|I_{mk}|} = \sin\left(\eta - \frac{\pi}{2}\right) = -\cos\eta \quad \longrightarrow \quad a(t) = \frac{1}{2}|I_{mk}|\,(1 - \cos\eta) \tag{4.82}$$

Using the fact that $dt = a d\eta$,

$$t = \int d\eta\, a = \frac{1}{2}|I_{mk}| \int d\eta\, (1 - \cos\eta) = \frac{1}{2}|I_{mk}|\,(\eta - \sin\eta) \tag{4.83}$$

This means that the scale factor for a closed, matter dominated universe is given by the parametric equations

$$a(t) = \frac{1}{2}|I_{mk}|\,(1 - \cos\eta), \quad t = \frac{1}{2}|I_{mk}|\,(\eta - \sin\eta), \quad I_{mk} = \frac{\Omega_m}{\Omega_k} \tag{4.84}$$

- Radiation dominated (Ω_γ):

$$\int dt = \frac{1}{H_0}\int da\, \frac{1}{a\left[\Omega_\gamma a^{-4} + \Omega_k a^{-2}\right]^{1/2}} = \int da\, \frac{1}{\left[|I_{\gamma k}|\, a^{-2} - 1\right]^{1/2}} \tag{4.85}$$

where $I_{\gamma k}$ is defined as before. Using the conformal time, we have that

$$\int d\eta = \int da\, \frac{1}{\left[|I_{\gamma k}| - a^2\right]^{1/2}} \quad \longrightarrow \quad \eta = \sin^{-1}\left(\frac{a}{|I_{\gamma k}|}\right) \tag{4.86}$$

Beyond Equilibrium

Let use consider what is occurring when these two species are converting into one another. The reaction can be categorised by some rate Γ, and it must compete against the expansion of the universe, with which we can associated the rate $H = \dot{a}/a$. The relative sizes of Γ and H dictate how important the reactions are in keeping the neutrons and protons equilibrated. One can write down a Boltzmann distribution for the comoving neutron number N_n, which is simply the number of neutrons within a comoving volume. This is given by

$$\boxed{\frac{d\log N_n}{d\log a} = -\frac{\Gamma}{H}\left[1 - \left(\frac{N_n^{\text{eq}}}{N_n}\right)^2\right]} \tag{4.120}$$

where $N_n^{\text{eq}} = a^3 n_n^{\text{eq}}$ is the equilibrium expression above. If $\Gamma \gg H$, then we have that $N_n \approx N_n^{\text{eq}}$, meaning that we can indeed use the equilibrium value predicted by (4.119). However, if $\Gamma \ll H$, then the expansion of the universe will dominate, and inhibit the depletion or creation of neutrons by that reaction. The equation is then

$$\frac{d\log N_n}{d\log a} \approx 0 \tag{4.121}$$

meaning that the comoving neutron number is *frozen out* (and the number density will decay as a^{-3}). Evidently, the transition between the regimes occurs at $\Gamma \approx H$, and will depend on how Γ depends on temperature and masses. It turns out that for this reaction, $k_B T_f \approx 0.8$ MeV. This means that the relative number density of neutrons to protons will be frozen in at $n_n^{\text{eq}}/n_p^{\text{eq}} \approx 1/6$. However, we observe something closer to $1/7$ due to the finite decay time of the neutron.

We can use a very simple argument to find the fraction of helium verses hydrogen in our universe. We initially have 7/8 in protons and 1/8 in neutron. Then we need to pair up the protons and neutrons reducing the number of unpaired protons to $6/8 \approx 75\%$. We thus expect to have roughly 25% of the mass in helium, and 75% in hydrogen.