$$= g(\underline{x}) - \frac{\langle v, b - Ax\rangle^2}{\langle v, Av\rangle}.$$

So for any vector $\underline{v} \neq 0$, we have $g(\underline{x} + \hat{t}v) < g(\underline{x})$ unless $\langle v, b - Ax\rangle = 0$, in which case $g(\underline{x}) = g(\underline{x} + \hat{t}v)$.

This gives the necessary condition because.

Suppose x^* satisfies $Ax^* = b$.

Then $\langle v, b - Ax^*\rangle = 0$ for any $\underline{v}$ and $g(\underline{x})$ cannot be made any smaller than $g(x^*)$.

Thus x^* minimizes g.

Conversely, suppose x^* minimizes g. Then for any vector v, we have

$$g(x^* + \hat{t}v) \geq g(x^*).$$

Thus $\langle v, b - Ax^*\rangle = 0 \implies b - Ax^* = 0$ or x^* is solution.

This set is linearly independent set and gives us search directions

$$t_k = \frac{\langle \underline{v}^{(k)}, b - A\underline{x}^{(k-1)} \rangle}{\langle \underline{v}^{(k)}, A\underline{v}^{(k)} \rangle} = \frac{\langle \underline{v}^{(k)}, \underline{r}^{(k-1)} \rangle}{\langle \underline{v}^{(k)}, A\underline{v}^{(k)} \rangle}$$

and $\underline{x}^{(k)} = \underline{x}^{(k-1)} + t_k \underline{v}^{(k)}$.

Theorem: Let $\{\underline{v}^{(1)}, \ldots, \underline{v}^{(n)}\}$ be an A-orthogonal set of nonzero vectors associated with the positive definite matrix A, and let $\underline{x}^{(0)}$ be arbitrary. Define

$$t_k = \frac{\langle v^{(k)}, \underline{b} - A\underline{x}^{(k-1)} \rangle}{\langle v^{(k)}, Av^{(k)} \rangle}$$

and $\underline{x}^{(k)} = \underline{x}^{(k-1)} + t_k \underline{v}^{(k)}$, for $k = 1, \ldots n$.

Then, assuming exact arithmetic $A\underline{x}^{(n)} = \underline{b}$.