Axiomatic approach to Probability Theory

Two important definitions:-

1) class of events: Just as an event is a collection of elementary events forms a class of events is a collection of events having some property(s) in common.
For ex: If A, B, C are events, then $\{A, B, C\}$ is a class of events.

2) σ-field of ~~the~~ events: It is a non-empty class of events which is closed under the formation of countable unions and complements. It is denoted by $\mathcal{A}$ or a.

$\mathcal{A}$ is non empty in the sense that there is at least one event A with respect to the random experiment under consideration such that $A \in \mathcal{A}$

$\mathcal{A}$ is closed under the formation of countable unions means that if

Date

$$\Rightarrow \bigcap_{i=1}^{\alpha} A_i \in \mathcal{A}$$

Theorem 4: $\mathcal{A}$ is closed under the formation of finite intersections.

Let $A_i \in \mathcal{A}$ $(i = 1, 2, \ldots n)$

Along with these events, we consider $A_{n+1}, A_{n+2}, \ldots$ such that $A_i = A_n$ $(i = n+1, n+2, \ldots)$

Since $\mathcal{A}$ is closed under the formation of countable intersections,

$$\bigcap_{i=1}^{\alpha} A_i \in \mathcal{A}$$

$$\left(\bigcap_{i=1}^{n-1} A_i\right) \cap \left(\bigcap_{i=n}^{\alpha} A_i\right) \in \mathcal{A}$$

$$\Rightarrow \left(\bigcap_{i=1}^{n-1} A_i\right) \cap (A_n \cap A_n \cap \ldots) \in \mathcal{A}$$

$$\Rightarrow \left(\bigcap_{i=1}^{n-1} A_i\right) \cap A_n \in \mathcal{A}$$

$$\Rightarrow \bigcap_{i=1}^{n} A_i \in \mathcal{A}$$

Theorem 5: $A \in \mathcal{A}, B \in \mathcal{A} \Rightarrow A - B \in \mathcal{A}$

$A \in \mathcal{A}, B^c \in \mathcal{A}$

$\Rightarrow A \cap B^c \in \mathcal{A}$

$\Rightarrow A - B \in \mathcal{A}$

Limitations of classical definition

1) Finite sample space: classical definition cannot be applied if the sample space w.r.t. a random exp. is not finite. For ex, classical def. fails in exp. like tossing of a coin until a head appears, where the sample space is countably infinite.

2) Equally likely outcomes: Even if the sample space is finite, classical definition cannot be applied directly if the outcomes of a random exp. are not equally likely. Thus the field of application of classical definition is limited only to such trivial experiments like tossing of a coin, rolling of a die etc., where one assumes that the coin or the die is unbiased.

3) Circularity: There is an inherent circularity in the classical def., for it is a def. of probability

Page No.
Date

Verification:-

Since p_i's are non-negative real nos., so their sum or the sum of a number of p_i's will be a non-negative real no.

$\therefore P(A) \geq 0$ for any $A \in \mathcal{A}$

$$P(\Omega) = \sum_{i:\omega_i \in \Omega} p_i = \sum_{i=1}^{\infty} p_i = 1$$

Let $A_j (j = 1, 2, \dots) \in \mathcal{A}$ be mutually exclusive events, then

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{i:\omega_i \in \left[\bigcup_{j=1}^{\infty} A_j\right]} p_i$$

$$= \sum_{i:\omega_i \in A_1} p_i + \sum_{i:\omega_i \in A_2} p_i + \dots$$

$$= P(A_1) + P(A_2) + \dots$$

$$= \sum_{j=1}^{\infty} P(A_j)$$

Thus P is seen to be a non-negative, normed and countably additive function. Thus P is a probability function defined on $\mathcal{A}$