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$$= P\left(\bigcup_{i=1}^{m} A_i\right) + P(A_{m+1}) - P\left(\left[\bigcup_{i=1}^{m} A_i\right] \cap A_{m+1}\right) \quad \ldots (5)$$

Again,

$$P\left(\left[\bigcup_{i=1}^{m} A_i\right] \cap A_{m+1}\right)$$

$$= P\left(\bigcup_{i=1}^{m} [A_i \cap A_{m+1}]\right)$$

$$= P\left(\bigcup_{i=1}^{m} B_i\right), \text{ say, where } B_i = A_i \cap A_{m+1} \; \forall i$$

$$= \sum_{i=1}^{m} P(B_i) - \sum_{\substack{i,j=1 \\ i<j}}^{m} P(B_i \cap B_j) + \sum_{\substack{i,j,k=1 \\ i<j<k}}^{m} P(B_i \cap B_j \cap B_k)$$

$$\ldots + (-1)^{m-1} P\left(\bigcap_{i=1}^{m} B_i\right)$$

[$\because$ We have already assumed an expression for $n = m$]

$$= \sum_{i=1}^{m} P(A_i \cap A_{m+1}) - \sum_{\substack{i,j=1 \\ i<j}}^{m} P(A_i \cap A_j \cap A_{m+1}) +$$

$$\sum_{\substack{i,j,k=1 \\ i<j<k}}^{m} P(A_i \cap A_j \cap A_k \cap A_{m+1}) - \cdots + (-1)^{m-1} P\left(\bigcap_{i=1}^{m+1} A_i\right) \quad \ldots (6)$$

Now by Boole's inequality,

$$P\left(\bigcup_{i=1}^{r} A_i^c\right) \leq \sum_{i=1}^{r} P(A_i^c)$$

$$\Rightarrow P\left(\bigcup_{i=1}^{r} A_i^c\right) \leq \sum_{i=1}^{r} \{1 - P(A_i)\}$$

$$\Rightarrow P\left(\bigcup_{i=1}^{r} A_i^c\right) \leq r - \sum_{i=1}^{r} P(A_i)$$

$$\Rightarrow -P\left(\bigcup_{i=1}^{r} A_i^c\right) \geq \sum_{i=1}^{r} P(A_i) - r$$

$$\Rightarrow 1 - P\left(\bigcup_{i=1}^{r} A_i^c\right) \geq \sum_{i=1}^{r} P(A_i) - (r-1)$$

$$\Rightarrow P\left(\bigcap_{i=1}^{r} A_i\right) \geq \sum_{i=1}^{r} P(A_i) - (r-1)$$

Hence the proof.

P-1 Arrange the following quantities in increasing order.
$P(A_1)$, $P(A_1)+P(A_2)$, $P(A_1)+P(A_2)-1$, $P(A_1 \cup A_2)$, $P(A_1 \cap A_2)$

$$-\cancel{P(A_1 \cup A_2) \leq P(A_1) + P(A_2)} \qquad P(A_1)+P(A_2)-1 = \sum_{i=1}^{2} P(A_i) - (2-1)$$

$$\cancel{P(A_1)+P(A_2)-1 \leq P(A_1) \leq P(A_1 \cup A_2)} < P(A_1 \cap A_2)$$

$$\cancel{\leq P(A_1) + P(A_2)}$$

$$P(A_1)+P(A_2)-1 < P(A_1 \cap A_2) < P(A_1) < P(A_1 \cup A_2) < P(A_1) + P(A_2)$$

Now,

$$B = \left(B \cap [A_{m+1} \cup A_{m+2} \cup \ldots \cup A_r]\right) \cup \left(B \cap [A_{m+1} \cup A_{m+2} \cup \ldots \cup A_r]^c\right)$$

$$\Rightarrow P(B) = P\left(B \cap [A_{m+1} \cup \ldots \cup A_r]\right) + P\left(B \cap [A_{m+1} \cup \ldots \cup A_r]^c\right)$$

[$\because$ The two events on RHS are mutually exclusive]

$$\Rightarrow P\left(B \cap [A_{m+1} \cup A_{m+2} \cup \ldots \cup A_r]^c\right) = P(B) - P\left(B \cap [A_{m+1} \cup A_{m+2} \cup \ldots \cup A_r]\right) \quad \ldots (2)$$

Again

$$P\left(B \cap [A_{m+1} \cup A_{m+2} \cup \ldots \cup A_r]\right)$$

$$= P\left([B \cap A_{m+1}] \cup [B \cap A_{m+2}] \cup \ldots \cup [B \cap A_r]\right)$$

$$= P\left(\bigcup_{i=1}^{r-m} [B \cap A_{m+i}]\right)$$

$$= \sum_{i=1}^{r-m} P(B \cap A_{m+i}) - \sum_{\substack{i,j=1 \\ i<j}}^{r-m} P(B \cap A_{m+i} \cap A_{m+j})$$

$$+ \sum_{i,j,k=1}^{r-m} P(B \cap A_{m+i} \cap A_{m+j} \cap A_{m+k})$$

$$- \ldots + (-1)^{r-m-1} P(B \cap A_{m+1} \cap A_{m+2} \cap \ldots \cap A_r)$$

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$$= \frac{(m+i)!}{i!\,m!} = \binom{m+i}{m} \quad , i = 0, 1, \dots, r-m$$

Thus,

$$P_{[m]} = S_m - \binom{m+1}{m} S_{m+1} + \binom{m+2}{m} S_{m+2} - \dots + (-1)^{r-m} \binom{r}{m} S_r$$

1) A person wrote n letters sealed them in n envelopes and wrote diff. addresses randomly on them. Show that the prob. that exactly m ($\leq n$) letters are placed in correct envelopes is approximately $\frac{1}{m!}$ when n is large.

— Let A_i be an event denoting that the ith letter will be put into the correct envelope ($i = 1, 2, \dots, n$). The required prob. is then given by

$$P_{[m]} = S_m - \binom{m+1}{m} S_{m+1} + \binom{m+2}{m} S_{m+2} - \dots + (-1)^{n-m} \binom{n}{m} S_n$$

where $S_1 = \sum_{i=1}^{n} P(A_i)$, $S_2 = \sum_{\substack{i,j=1 \\ i<j}}^{n} P(A_i \cap A_j)$,

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$$S_3 = \sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} P(A_i \cap A_j \cap A_k)$$

The total no. of ways n letters may be put into n envelopes is $n!$. If the ith letter is put into the correct envelope then the remaining $n-1$ letters may be placed into the other $n-1$ envelopes in $(n-1)!$ ways.

$$\therefore P(A_i) = \frac{(n-1)!}{n!} \quad \forall i$$

$$\therefore P(A_i \cap A_j) = \frac{(n-2)!}{n!} \quad \forall i<j$$

$$\therefore P(A_i \cap A_j \cap A_k) = \frac{(n-3)!}{n!} \quad \forall i<j<k$$

and so on.

Proceeding in this way, we may say that each term of S_m has the value $\frac{(n-m)!}{n!}$ and there are in all $\binom{n}{m}$ terms. Each term of S_{m+1} has the value $\frac{(n-m-1)!}{n!}$ and there are in all $\binom{n}{m+1}$ terms and so on. Hence in general, each term of S_{m+i} has the value $\frac{(n-m-i)!}{n!}$ and there are $\binom{n}{m+i}$ terms in all. $(i = 0, 1, 2, n-m)$