Ans.ctd.

Since $f(x,y) = x^2y$ and $g(x,y) = x$, it follows that

$$\int_C x^2ydx + xdy = \iint_R [\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(x^2y)]dA$$

$$= \int_0^1 \int_0^{2x} (1 - x^2)dydx$$

$$= \int_0^1 (2x - 2x^3)dx$$

$$= \left[x^2 - \frac{x^4}{2}\right]_0^1 = \underline{\underline{\frac{1}{2}}}$$

Ans.

Since $f(x,y) = (xy + y^2)$ and $g(x,y) = x^2$, it follows that

$$\int_C (xy + y^2)dx + x^2 dy = \iint_R [\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)]dA$$

$$= \iint_R [(2x) - (x + 2y)]dxdy$$

$$= \iint_R (x - 2y)dxdy$$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)dydx$$

$$= \int_0^1 \left[xy - y^2\right]_{x^2}^{x} = \int_0^1 (x^4 - x^3)dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \underline{\underline{\frac{-1}{20}}}$$

Ans.

The work W performed by the field is

$$W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (e^x - y^3)\,dx + (\cos y + x^3)\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(\cos y + x^3) - \frac{\partial}{\partial y}(e^x - y^3) \right] dA \quad \text{By Green's theorem}$$

$$= \iint_R (3x^2 + 3y^2)\,dA = 3 \iint_R (x^2 + y^2)\,dA$$

$$= 3 \int_0^{2\pi} \int_0^1 (r^2) r\,dr\,d\theta = \frac{3}{4} \int_0^{2\pi} d\theta = \frac{3\pi}{2}$$

converted to polar coordinates.

Qn.9

Use a line integral to find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Gauss Divergence Theorem

Theorem

Let $F = fi + gj + hk$ be a vector field with the functions f, g, h having continuous first order partial derivaives at all points in a solid region H having volume V whose surface S is closed and bounded. Then

$$\iint_S F.\hat{\mathbf{n}}dS = \iiint_V \nabla.FdV$$

where $\hat{\mathbf{n}}$ is the outward unit normal to the surface S.

Ans.ctd.

On $S_3, y = 0, F = x^2 i - xzj + z^2 k, \hat{\mathbf{n}} = -j, F.\hat{\mathbf{n}} = xz$

$$\iint_{S_3} F.\hat{\mathbf{n}} dS_3 = \int_0^a \int_0^c xz dz dx = \frac{a^2 c^2}{4}$$

On $S_4, y = b, F = (x^2 - bz)i + (b^2 - xz)j + (z^2 - bx)k, \hat{\mathbf{n}} = j, F.\hat{\mathbf{n}} = b^2 - xz$

$$\iint_{S_4} F.\hat{\mathbf{n}} dS_4 = \int_0^a \int_0^c (b^2 - xz) dz dx = acb^2 - \frac{a^2 c^2}{4}$$

Ans.ctd

On $S_5, x = 0, \vec{F} = z\vec{j} + yz\vec{k}, \hat{n} = -\vec{i}, \vec{F}.\hat{n} = 0$

$$\int\int_{S_5} \vec{F}.\hat{n}\, dS_5 = 0$$

On $S_6, x = 1, \vec{F} = \vec{i} + z\vec{j} + yz\vec{k}, \hat{n} = \vec{i}, \vec{F}.\hat{n} = 1$

$$\int\int_{S_6} \vec{F}.\hat{n}\, dS_6 = \int_0^1 \int_0^1 1\, dydz = 1$$

$$\begin{aligned}
\int\int_{S} \vec{F}.\hat{n}\, dS &= 1 + \frac{1}{2} \\
&= \frac{3}{2}
\end{aligned}$$

$$\text{div}\vec{F} = 1 + y$$

$$\begin{aligned}
\int\int\int_V \text{div}\vec{F}\, dV &= \int\int\int_V 1 + y\, dV \\
&= \int_0^1 \int_0^1 \int_0^1 1 + y\, dz\, dy\, dx \\
&= \int_0^1 \int_0^1 1 + y\, dy\, dz \\
&= \int_0^1 \left[y + \frac{y^2}{2} \right] dx \\
&= \frac{3}{2} \int_0^1 dx \\
&= \frac{3}{2}
\end{aligned}$$

Qn. 5

Problem 6.5.25. *Use divergence theorem to evaluate* $\int\int_S \vec{F}.\hat{n}dS$ *where* $\vec{F} = 5\vec{i} + 10y\vec{j} + z^3\vec{k}$ *and* S *is the surface of* $-1 \le x \le 1, \frac{1}{2}x \le y \le x, 0 \le z \le y$

Solution:

$$\begin{aligned}
\int\int\int_V \mathrm{div}\vec{F}\, dV &= \int\int\int_V (10 + 3z^2)\, dV \\
&= \int_{-1}^{1}\int_{x/2}^{x}\int_0^y (10 + 3z^2)\ dz\, dy\, dx \\
&= \int_{-1}^{1}\int_{x/2}^{x} (10y + y^3)\, dy\, dx \\
&= \int_{-1}^{1} \left(5y^2 + \frac{y^4}{4}\right)_{x/2}^{x} dx \\
&= \int_{-1}^{1} \left(\frac{15x^2}{4} + \frac{15x^4}{644}\right) dx \\
&= \frac{83}{32}
\end{aligned}$$

Ans.

Stoke's theorem is

$$\int_C \vec{F}\cdot\vec{r} = \int\int_S \text{curl}\vec{F}\cdot\hat{n}\,dS$$

Here $\vec{F} = (x+y)\vec{i} + (2x-y)\vec{j} + (y+z)\vec{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y) & (2x-y) & (y+z) \end{vmatrix}$$

$$= \vec{i} + \vec{k}$$

$$\hat{n} = \text{unit vector normal to the surface}$$

$$= \vec{k}$$

$$\therefore \text{curl}\vec{F}\cdot\hat{n} = 1$$

$$\text{Also } dS = dydx$$

$$\therefore \int\int_S \text{curl}\vec{F}\cdot\hat{n}\,dS = \int\int_R dy\,dx$$

$$= \text{Area of the triangular region region R}$$

$$= \frac{1}{2}\times 2\times 3 = 3$$

Qn.3

Use Stoke's theorem to evaluate $\iint_S curlF.\hat{\mathbf{n}}dS$ where S is the part of the surface of the paraboloid $z = 1 - x^2 - y^2$ for which $z \geq 0$ and $F = yi + zj + xk$

Problems

- Use Stoke's theorem to evaluate $\oint_C F.dr$, where $F(x, y, z) = 3zi + 4xj + 2yk$, where C is the boundary of the paraboloid $z = 4 - x^2 - y^2$