Dedication

I am deeply grateful to my ***Lord***, the Master planner of my life. The gratefulness goes also to ***Prof C. Chidume*** the Acting President of the African University of Science and Technology. I would like to thank also my supervisor ***Dr Guy Degla*** for having introduced me to the study of Galerkin Method and for all the time, the great patience, and the attention he devoted to my work. The opportunity to learn from his generosity and his teachings (about mathematics not only) was at the same time stimulating and pleasant.

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But my first thought and my warmest thank goes to ***Bolaji:*** "let me dedicate this to you ...".
Thanks!

This means that E is a completion of X if E is a Banach space which contains a dense subset isometric to X.

Theorem 1.1.11 (Hausdorff)

Every normed linear space has a completion.

Definition 1.1.12

Let $(X, ||\cdot||_{X})$ and $(Y, ||\cdot||_{Y})$ be two arbitrary normed linear spaces. A map $f : X \to Y$ is said to be continuous at a point $a \in X$ if for every sequence $(x_n)_n$ of X converging to a with respect to $||\cdot||_{X}$, the sequence $\big(f(x_n)\big)_n$ converges to $f(a)$ in Y with respect to $||\cdot||_{Y}$. f is said to be continuous (on X) if it is continuous at every point of X.
Equivalently, f is continuous if and only if the pre-image of every open set in Y is an open set in X.

Recall that given two $\mathbb{K}$-linear spaces X and Y, a map or operator

$$T : X \longrightarrow Y$$

is said to be *linear* if for all $x_1, x_2 \in X$ and for all $\alpha_1, \alpha_2 \in \mathbb{K}$ we have

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2).$$

Equivalently, $T : X \longrightarrow Y$ is linear if for all $x_1, x_2 \in X$ and for all $\alpha \in \mathbb{K}$, we have

$$T(\alpha x_1 + x_2) = \alpha T(x_1) + T(x_2).$$

Theorem 1.1.13

Let $(X, ||\cdot||_{X})$ and $(Y, ||\cdot||_{Y})$ be normed linear spaces. Then a linear map $T : X \to Y$ is continuous if and only if T is a *bounded linear map* in the sense that there exists a constant real number $\alpha \geq 0$ such that

$$||T(x)||_{Y} \leq \alpha\, ||x||_{X} \quad \forall x \in X.$$

Notations 1.1.14
Let X and Y be two given arbitrary normed linear spaces.

The set of all bounded linear maps (i.e. continuous linear maps) from X into Y is a linear space that will be denoted by $\mathcal{B}(X, Y)$.

Given a bounded linear map $T : X \to Y$, we shall set

$$||T||_{\mathcal{B}(X,Y)} = \inf\Big\{\, k : \ ||T(x)||_{Y} \leq k||x||_{X} \ \forall x \in X \,\Big\}$$

that will be simply written as $||T||$ when there is no ambiguity.

We denote by $X^* := \mathcal{B}(X, \mathbb{K})$ the topological *dual* of X; that is, the set of all continuous linear functionals of X.

Proposition 1.1.15

Let $(X, ||\cdot||_X)$ be a nontrivial normed linear space and $(Y, ||\cdot||_Y)$ be an arbitrary normed linear space. Then for every $T \in \mathcal{B}(X, Y)$, we have

$$||T(x)||_Y \leq ||T||\,||x||_X \quad \forall\, x \in X\,,$$

and

$$||T|| = \sup_{||x||_X \leq 1} ||T(x)||_Y = \sup_{||x||_X = 1} ||T(x)||_Y = \sup_{||x||_X \neq 0} \frac{||T(x)||_Y}{||x||_X}\,.$$

Theorem 1.1.16

Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be normed linear spaces. Then

1. $\big(\mathcal{B}(X, Y)\,,\, ||\cdot||_{\mathcal{B}(X,Y)}\big)$ is a normed linear space.
2. If moreover $(Y, ||\cdot||_Y)$ is a Banach space, then $\big(\mathcal{B}(X, Y),\, ||\cdot||_{\mathcal{B}(X,Y)}\big)$ is a Banach space.

Corollary 1.1.17

The dual X^* of any normed linear space X is (always) a Banach space.

Remark 1.1.18

Given a normed linear space $(X, ||\cdot||_X)$, the dual X^* being a normed linear space (in fact a Banach space) has also a dual X^{**} called the *bidual* of X. Moreover there exists a canonical injection $J : X \hookrightarrow X^{**}$ defined by

$$\begin{array}{rccl} J\,: & X & \longrightarrow & X^{**} \\ & x & \longmapsto & J(x)\,, \end{array}$$

where $J(x)$ the continuous form on X^* defined by

$$\langle J(x),\, f\rangle := \langle f,\, x\rangle := f(x)\;;\quad \forall\, f \in X^*\,.$$

Definition 1.1.19 (Reflexive space)

A normed linear space $(X, ||\cdot||_X)$ is *reflexive* if it is a Banach space such that the canonical injection $J : X \hookrightarrow X^{**}$ is surjective.

where

$$N_p(u) \;=\; \begin{cases} \int_\Omega |u|^p dx)^{\frac{1}{p}} & \text{if} \quad 1 \le p < \infty, \\ \\ \inf\{M \ge 0 \,:\; |u(x)| \le M \text{ a.e.}\} & \text{if} \quad p = \infty\,. \end{cases}$$

(iv) ${C_c}^\infty(\Omega)$ denote the space of infinitely many times differentiable functions $u : \Omega \longrightarrow \mathbb{R}$ with compact support in Ω.
(v)
(a)

$$\mathscr{D}(\Omega) \;=\; \Big\{ u \in C^\infty(\Omega);\; \text{supp}(u) \text{ is compact and } \text{supp}(u) \subset \Omega \Big\} \;=\; {C_c}^\infty(\Omega)$$

is generally called the *set of tests functions.*
(b) $\mathscr{D}(\bar\Omega)$ is the space of all functions v such that v is the restriction on $\overline{\Omega}$ of a function of $\mathscr{D}(\mathbb{R}^n)$.
(vi) The space of locally integrable functions is denoted by

$${L^1}_{loc}(\Omega) \;=\; \bigcap_{K \subset\subset \Omega} L^1(K),$$

where K is a compact subset of Ω.
(vii)
(a)

$$C^k(\Omega) \;=\; \Big\{ u : \Omega \longrightarrow \mathbb{R},\; u \text{ is } k\text{-times continuously differentiable} \Big\}$$

$$C^k(\bar\Omega) \;=\; \Big\{ u \in C^k(\Omega);\; D^\alpha(u) \text{ is uniformly continuous for all } |\alpha| \le k. \Big\}\,.$$

$$C^\infty(\bar\Omega) \;=\; \bigcap_{k=0}^{\infty} C^k(\bar\Omega)\,.$$

Thus if $u \in C^k(\bar\Omega)$ then D^αu continuously extends to $\bar\Omega$ for each multi-index α, $|\alpha| \le k$.

Definition 1.4.1 (Weak derivative)

Let $u, v \in {L^1}_{loc}(\Omega)$ and α is a multi-index.
We say that v is the α^{th}-*weak derivative* of u and write $D^\alpha u = v$ if

$$\int_\Omega u\, D^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega v\, \phi dx,$$

for all $\phi \in \mathscr{D}(\Omega)$.

CHAPTER 2

Galerkin Method

2.1 Analysis of PDEs.

Partial Differential Equations (PDE's) are fundamental in many areas of Mathematics such as Differential Geometry and Stochastic Processes. Nowadays many natural human or biological, chemical, mechanical, economical or financial systems and processes can be described at a macroscopic level by a set of PDE's governing averaged quantities such as density, temperature, concentration, velocity, etc.

As there is no general theory known for solving all PDE's, and given the variety of phenomena modelled by such equations, research focuses on particular PDE's that are important for theory or applications. For example, for PDE's of order 2; *elliptic* equations are associated to a special state of a system in principle corresponding to the minimum of the energy, *parabolic* problems describe evolutionary phenomena that lead to a steady state described by an elliptic equation, *hyperbolic* equations modelled the transport of some physical quantity such as fluids or waves.

Thus in our dissertation, we would like to present a constructive method for solving Boundary Value Problems (i.e., PDE's subjected to Boundary Conditions) of variational type that have the variational formulation :

$$(P) \qquad \begin{cases} \text{Find} \quad u \in H \quad \text{such that,} \\ \mathcal{A}(u,v) \; = \; \mathcal{L}(v), \quad \forall v \in H \end{cases}$$

where H is an infinite dimensional Hilbert space, $\mathcal{L} : H \longrightarrow \mathbb{R}$ is a bounded linear form and $\mathcal{A} : H \times H \longrightarrow \mathbb{R}$ is coercive and continuous bilinear form.

V. Then there exists a sequence $(w_n)_n \subset W$ such that $v_n = Aw_n$ for every n. Moreover for all m, n, we have

$$||w_n - w_m|| \;\leq\; \frac{||A(w_n - w_m)||}{\alpha} \;=\; \frac{||v_n - v_m)||}{\alpha}$$

by the inequality (2.5) and the linearity of A. Therefore $(w_n)_n$ is a Cauchy sequence and so converges in the Hilbert W to some element w. It follows from the continuity of A that $w = Av$. This completes the proof of the closedness of ImA.
Now let $v_o \in (\mathrm{Im}A)^{\perp}$. Then $\langle v_o,\, v\rangle_V = 0$ for all $v \in \mathrm{Im}A$ which means also that

$$\langle v_o,\, Aw\rangle_V \;=\; \langle Aw,\, v_o\rangle_V \;=\; 0\,, \qquad \forall\, w \in W.$$

And so

$$\mathcal{A}(w, v_o) \;=\; 0\,, \qquad \forall\, w \in W\,.$$

Therefore $v_o = 0$ according to (2.3).
Hence A is a continuous linear bijection with a continuous inverse (cf. the inequality (2.5)).
Besides, by applying Riesz Representation Theorem to $\mathcal{L} \in V^*$, there exists $\bar{v} \in V$ such that

$$\mathcal{L}(v) \;=\; \langle v,\, \bar{v}\rangle_V\,, \qquad \forall v \in V, \qquad \text{and} \quad ||\mathcal{L}||_{V^*} \;=\; ||\bar{v}||_V\,.$$

By setting $\bar{v} = B(\mathcal{L})$, it is clear that B defines a linear continuous and isometric map from V^* onto V.

Therefore $(\mathscr{P})$ is reduced to finding, for any given $\mathcal{L} \in V^*$, an element $u \in W$ such that

$$Au \;=\; B(\mathcal{L})\,. \tag{2.6}$$

Thus we have a unique solution

$$u \;=\; A^{-1}\big(B(\mathcal{L})\big)$$

that satisfies moreover

$$\begin{aligned} ||u||_W \;&\leq\; \frac{||Au||_V}{\alpha} \\ &=\; \frac{||B(\mathcal{L})||_V}{\alpha} \\ &=\; \frac{||\mathcal{L}||_{V^*}}{\alpha}\,. \end{aligned}$$

$\square$

Let us now consider the particular case in which $V = W$ in problem $(\mathscr{P})$.

Let us show that

$$V_h \;=\; \Big\{ v_h \in C([a,b]);\; v_{h|_{K_i}} \in \mathbb{P}_1,\; \forall i \in \{1,\ldots,N\} \Big\} \bigcap V$$

is a finite dimensional subspace of $V = H_0^1(a,b)$. It suffices to show that the functions defined above constitute a basis for V_h. First of all, these functions are continuous on each intervals $K_i = [x_{i-1}, x_i]$ as polynomials, and piecewise linear and they vanish on $\{a,b\}$, so $\phi_i \in V_h$. Observe also that $\phi_i(x_j) = \delta_{ij}$. Let $\{\alpha_i\}_{i=1}^{N-1} \in \mathbb{R}$, such that $f(x) = \sum_{i=1}^{N-1} \alpha_i \phi_i(x) = 0$, $\forall x \in [a,b]$. Therefore

$$f(x_1) = \alpha_1 = 0,\ \ldots,\ f(x_{N-1}) = \alpha_{N-1} = 0$$

Hence, $\{\phi_i\}_{i=1}^{N-1}$ are linearly independent. Furthermore, any $v_h \in V_h$ is uniquely written (because a polynomial of degree 1 on an interval $[c,d]$ is uniquely determined by its values on c and d) by $v_h = \sum_{i=1}^{N-1} \beta_i \phi_i$. This implies that

$$v_h(x_1) = \beta_1,\ \ldots,\ v_h(x_{N-1}) = \beta_{N-1}\ .$$

This identity shows that $\{\phi_i\}_{i=1}^{N-1}$ is a basis for V_h. Therefore

$$V_h = \operatorname{span}\Big\{ \phi_i\,,\ 1 \le i \le N-1 \Big\} \quad \Longrightarrow \quad \dim V_h = N-1$$

Note that the support of ϕ_i is

$$\operatorname{supp}\phi_i \;=\; [x_{i-1}, x_{i+1}], \quad i \in [1, N-1].$$

(b) Consider a two dimensional polygonal domain, covered with finite element meshes τ_h such that each element $K_e \in \tau_h$ is a triangle. So let K_e be a triangle with vertices (x_i, y_i),(x_j, y_j) and (x_k, y_k) taken in the anti-clockwise direction. We write a linear approximation inside each element of the form

$$u^{(K_e)}(x,y) = a_1 + a_2 x + a_3 y \qquad (\star)$$

with $a_i \in \mathbb{R}$. At the nodes we get
$u^{(K_e)}(x_i, y_i) = u_i = a_1 + a_2 x_i + a_3 y_i$,
$u^{(K_e)}(x_j, y_j) = u_j = a_1 + a_2 x_j + a_3 y_j$,
$u^{(K_e)}(x_k, y_k) = u_k = a_1 + a_2 x_k + a_3 y_k$.
For the solution of a_1, a_2, a_3 we have the following system

$$\begin{pmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} u_i \\ u_j \\ u_k \end{pmatrix},$$

using Cramer's rule, the solution of the system is obtained as

$$a_1 = \frac{\Delta_1}{\Delta}, \quad a_2 = \frac{\Delta_2}{\Delta}, \quad a_3 = \frac{\Delta_3}{\Delta}$$

Theorem 3.3.10

Let τ_h be a regular triangulation of Ω containing only triangles if $n = 2$ or tetrahedrals if $n = 3$. Let us denote by

$$\mathcal{N}_h \;=\; \left\{ c_i : \;\; i = 1, ..., N_h \right\}$$

the set of nodals in the mesh satisfying hypothesis (H_0), (H_1), (H_2).
Then there exists a basis system $\phi_i \; \forall i = 1, ..., N_h$ defined by

$$\begin{cases} \phi_{i|_K} \in P_m, & \forall \varphi \in \tau_h, \\ \phi_i(c_j) = \delta_{ij}, & \forall j = 1, ..., N_h, \end{cases}$$

called *shape functions* such that for all $v_h \in V_h^m$,

$$v_h = \sum_{i=1}^{N_h} v(c_i)\phi_i$$

How to construct shape functions?

It is appropriate to use reference element technique. It is particularly suitable for higher dimensional problems. When $n = 1$, it consists of computing a shape funtion on, a suitably choosen reference element say K_a. For each element K_i in the mesh we define then an affine reference map $F_{K_i} : K_a \longrightarrow K_i$ and use it to transfer the shape functions from K_a to K_i. In this way one obtains the desired finite element basis in the physical mesh τ_h.

Example 3.3.11

Suppose that the triangulation τ_h contains only triangles. Let us choose as reference element the triangle T_r with vertices $t_1 = (0, 0)$, $t_2 = (1, 0)$, $t_3 = (0, 1)$. Shape functions $\phi_{i,r}$ $i = 1, 2, 3$, are given by the following barycentric coordinates functions λ_i $i = 1, 2, 3$

$$\begin{aligned} \phi_{1,r}(x, y) &= 1 - x - y = \lambda_1(x, y), \\ \phi_{2,r}(x, y) &= x = \lambda_2(x, y), \\ \phi_{3,r}(x, y) &= y = \lambda_3(x, y), \end{aligned}$$

for all $(x, y) \in T_r$.

Proof : From **theorem 3.3.12** we have seen that for all $\alpha_i \in \mathbb{R}$ $i = 1, 2, 3$, there exists a unique $p \in \mathbb{P}_1$ such that $p(t_i) = \alpha_i$, $i = 1, 2, 3$ with $p(x, y) = \alpha_1\lambda_1(x, y) + \alpha_2\lambda_2(x, y) + \alpha_3\lambda_3(x, y)$ $\forall x, y \in T_r$. On the other hand by assuming $p(x, y) = a + by + cx$, $a, b, c \in \mathbb{R}$, $\forall x, y \in T_r$ we obtain a unique solution $c = \alpha_2 - \alpha_1$, $b = \alpha_3 - \alpha_1$, and $a = \alpha_1$ from $p(t_i) = \alpha_i$. So $p(x, y) = (\alpha_2 - \alpha_1)x + (\alpha_3 - \alpha_1)y + \alpha_1 = (1 - x - y)\alpha_1 + x\alpha_2 + y\alpha_3$. Thus for $(\alpha_1, \alpha_2, \alpha_3) = (1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ we have respectively by unicity of p

$$p_1(x,y) = 1 - x - y = \lambda_1(x,y)$$
$$p_2(x,y) = x = \lambda_2(x,y)$$
$$p_3(x,y) = y = \lambda_3(x,y)\ .$$

$\forall x, y \in T_r$. Which constitute a basis on T_r from **Theorem 3.3.4** called shape functions.

The transfer affine map which transfer T_r to a given triangle $T \in \tau_h$, with vertices $(t_1 = (x_1, y_1), t_2 = (x_2, y_2), t_3 = (x_3, y_3))$, is then defined by,

$$\begin{aligned} F :\ & T_r \longrightarrow T \\ & (s,t) \longmapsto (x,y) = F(s,t) = \begin{pmatrix} (x_2 - x_1)s + (x_3 - x_1)t + x_1 \\ (y_2 - y_1)s + (y_3 - y_1)t + y_1 \end{pmatrix}. \end{aligned}$$

where F is fixed such that $F(0,0) = t_1$, $F(1,0) = t_2$, $F(0,1) = t_3$,

In general, the stiffness matrix and the load vector are not easy to compute exactly. In such cases one use *numerical quadrature methods*. Among the wide scale of existing numerical quadrature methods, the *Gaussian quadrature rules* are of high efficiency.

3.4 Gaussian Quadrature Rules

One dimensional case

Definition 3.4.1:

The $(m+1)$-point Gaussian quadrature rule in the interval $K_a = (-1,1)$ has the form

$$\int_{-1}^{1} g(\xi)d\xi \approx \sum_{i=0}^{m} w_{m+1,i}\, g(\xi_{m+1,i}),$$

where g is a real bounded continuous function on $[-1,1]$, $\xi_{m+1,i} \in (-1,1)$, $i = 0, ..., m$, are the integration points, and $w_{m+1,i} \in \mathbb{R}$ are the integration weights which satisfies

$$\sum_{i=0}^{m} w_{m+1,i} = 2.$$

Definition 3.4.2 (Legendre polynomial):

Let the integer $m \geq 0$. Polynomials of the form

$$L_m(x) = \tfrac{1}{2^m m!}\tfrac{d^m}{dx^m}(x^2-1)^m,$$

. Then

$$\int_{-1}^{1} p(x)dx = \sum_{i=0}^{m} p(x_{m+1,i}) w_{m+1,i} \qquad \forall p \in \mathbb{P}_{2m}(-1,1) \qquad .$$

In order to have the Gauss-Radau formula to include the point $x = 1$, the variable a is taken in such a way that $q(1) = 0$ and a similar result as the previously presented is valid.

Gauss-Lobatto-Legendre quadrature formula:

Finally, the Gauss-Lobatto-Legendre quadrature formula is obtained by considering

$$q(x) = L_{m+1}(x) + aL_m(x) + bL_{m-1}, \qquad (c)$$

where a and b are chosen such that $q(-1) = q(1) = 0$.

Let $x_0 < x_1 < ... < x_m$ be the roots of (c) and let $w_0, ..., w_m$ be the solution of the linear system

$$\sum_{i=0}^{m} x_i^j w_i = \int_{-1}^{1} x^j dx, \qquad 0 \leq j \leq m$$

Then

$$\int_{-1}^{1} p(x)dx = \sum_{i=0}^{m} p(x_{m+1,i}) w_{m+1,i} \qquad \forall p \in \mathbb{P}_{2m-1}(-1,1).$$

Remark 3.4.4

The integration points and the integration weight exist and they are unique since they are roots of the Legendre polynomial. Furthermore we have the relation

$$w_{m+1,i} = \frac{2}{(1-\xi_{m+1,i}^2) L'_{m+1}(\xi)^2}, \qquad i = 0, ...m.$$

Quadrature in arbitrary intervals

Let $K = (x_{i-1}, x_i) \subset \mathbb{R}$ be an arbitrary interval. To transfer data from K_a to K we use an affine map $F_K : K_a \longrightarrow K$, such that

$$F_K(\xi) = c_1 + c_2\xi, \text{ for some } c_1, c_2 \in \mathbb{R},$$
$$F_K(-1) = x_{i-1},$$
$$F_K(1) = x_i.$$

Therefore, the new integration points $\tilde{\xi}_{m+1,i} \in K$ are then defined as

$$\tilde{\xi}_{m+1,i} = F_K(\xi_{m+1,i}), \; i = 0, ..., m.$$

Thus, K is the following matrix :

$$\begin{pmatrix} \frac{h_0}{3} + \frac{1}{h_0} & \frac{h_0}{6} - \frac{1}{h_0} & 0 & 0 & \dots & 0 \\ \frac{h_0}{6} - \frac{1}{h_0} & \frac{h_0+h_1}{3} + \frac{1}{h_0} + \frac{1}{h_1} & \frac{-h_1}{3} - \frac{1}{h_1} & 0 & \dots & \vdots \\ 0 & \frac{-h_1}{3} - \frac{1}{h_1} & \frac{h_1+h_2}{3} + \frac{1}{h_1} + \frac{1}{h_2} \frac{-h_2}{3} - \frac{1}{h_2} & 0 & & \\ 0 & 0 & \frac{-h_2}{3} - \frac{1}{h_2} & \frac{h_2+h_3}{3} + \frac{1}{h_2} + \frac{1}{h_3} & \frac{-h_3}{3} - \frac{1}{h_3} & 0 \\ 0 & \dots & 0 & \frac{-h_{N-1}}{3} - \frac{1}{h_{N-1}} & \frac{h_{N-1}+h_N}{3} + \frac{1}{h_{N-1}} + \frac{1}{h_N} & \frac{-h_N}{3} - \frac{1}{h_N} \\ 0 & \dots & 0 & 0 & -\frac{h_N}{3} - \frac{1}{h_N} & \frac{h_{N+1}}{3} + \frac{1}{h_{N+1}} \end{pmatrix}$$

Which is a tridiagonal matrix, and it is obviously sparse.