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CHAPTER 1

Preliminary notions

The idea behind weak and weak star topology [**3**, Chapter 3] is based on the construction of a topology on a set E with *minimum* open sets which makes continuous the elements $f_i : E \longrightarrow X_i$ of a given family $\{f_i\}_{i\in I}$ of maps, where $\{X_i\}_{i\in I}$ is a family of topological spaces, and I is an arbitrary index set.

Let E be a Banach space. We denote by E^* the set of linear and continuous maps from E to $\mathbb{R}$. For each $f \in E^*$, we associate a map $\phi_f : E \longrightarrow \mathbb{R}$ defined by

$$\phi_f(x) := \langle f, x\rangle = f(x)$$

for all $x \in E$.

Definition 1.1. *The weak topology on E is the coarsest topology on E which makes all maps ϕ_f continuous.*

Theorem 1.2. *Let K be a closed and convex subset of E. Then K is closed in the weak topology.*

Proof. See [**3**, Theorem 3.7]. $\square$

For each $x \in E$, we associate the map $\phi_x : E^* \longrightarrow \mathbb{R}$ defined by

$$\phi_x(f) := \langle f, x\rangle = f(x).$$

Definition 1.3. *The weak star topology is the coarsest topology on E^* for which all maps ϕ_x are continuous.*

Notice that on E^* we can consider both the weak topology and the weak star topology.

Remark 1.4. (i) The weak and the weak star topologies are Hausdorff. See [**3**, Proposition 3.3 and Proposition 3.11].

(ii) If E is of finite dimensional, then the strong and the weak topologies coincide. Identifying E with E^*, also the weak star topology coincides with the strong topology.

Proposition 1.5. *Let $\{f_n\} \subset E^*$ be a sequence. Then*

- *$f_n \rightharpoonup f$ in the weak star topology iff $\langle f_n, x\rangle \longrightarrow \langle f, x\rangle$ for all $x \in E$;*
- *$f_n \rightharpoonup f$ in the weak topology $\Longrightarrow f_n \rightharpoonup f$ in the weak star topology;*
- *if $f_n \rightharpoonup f$ in the weak star topology and $x_n \longrightarrow x$ strongly in E then $\langle f_n, x_n\rangle \longrightarrow < f, x >$;*
- *$f_n \rightharpoonup f$ in the weak star topology $\Longrightarrow \{f_n\}$ is bounded and $\|f\| \le \liminf_{n\to+\infty} \|f_n\|$.*

Assume that $\mathcal{A}$ is a bounded subset of the normed vector space $(E, \|.\|)$. Then there exists a positive constant c such that $\|v\| \leq c$ for all admissible controls v. In this case, the sequence is uniformly bounded, that is, $\|\alpha_k\| \leq c$ for all k. If E is a Hilbert space, by the Banach Alaoglu theorem, $\{\alpha_k\}_{k\in\mathbb{N}}$ has a subsequence that weakly converges in E. If we denote by the subequence $\{\alpha_k\}_{k\in\mathbb{N}}$ again, the convergence $\alpha_k \longrightarrow \alpha$ means that the scalar products $< \alpha_k, \lambda >$ converges to $< \alpha, \lambda >$ for every function $\lambda \in E$.

We have established so far that there is a weak limit of the minimizing sequence. But it is not clear if this limit belongs to the set of admissible controls. Suppose $\mathcal{A}$ is convex and closed (and consequently contains the limits of every sequence of its elements that converges in the norm). It is known from the theorem of Hilbert spaces that every convex closed subset of a Hilbert space is weakly convex, that is, it contains the limits of all weakly converging sequences of its elements. Since the minimizing sequence consists only of the elements of E and weakly converges, it follows that its weak limit α belongs to $\mathcal{A}$ and is therefore an admissible control. However, it is not known whether the functional P achieves its lower bound at α.

Assume that the functional P is convex and continuous. Every convex continuous functional is weakly lower semicontinuous. This means that if $\alpha_k \longrightarrow \alpha$ weakly in E, then

$$P(\alpha) \leq \inf \lim P(\alpha_k). \tag{2.6}$$

This inequality implies that the sequence $\{P(\alpha_k)\}_{k\in\mathbb{N}}$ has converging subsequences, although it does not necessarily converge itself (which follows from the Bolzano-Weierstrass theorem and the Banach Alaogulu theorem)[4]. It follows from 2.6, since the functional is weakly semicontinuous, $P(\alpha)$ does not exceed the lower bound of limits of all subsequences of $\{P(\alpha_k)\}_{k\in\mathbb{N}}$.

Since the subject of our consideration is not an arbitrary weakly converging saequence, but the one that minimizes the functional P, $\{P(\alpha_k)\}_{k\in\mathbb{N}}$ not only has converging subsequences, but converges itself to the lower bound of the functional P on $\mathcal{A}$. Then inequality 2.6 can be written in the form

$$P(\alpha) \leq \inf \lim P(\alpha_k) = \inf P(\mathcal{A}),$$

which means that the value of the functional P at the element α does not exceed its lower bound on the set $\mathcal{A}$. We establised earlier that this element belongs to $\mathcal{A}$.

Since none of the elements of a set of numbers can be less than its lower bound, the foregoing relation turns out to be an equality. The value of the functional at the element $\alpha \in \mathcal{A}$ is equal to its lower bound on $\mathcal{A}$. Thus, the admissible control α is a solution to the problem in question.

[4]See Brezis in functional analysis part.

THEOREM 3.1 (**First-Order Necessary Conditions**). *Consider the problem to minimize the functional*

$$P(\alpha) = \int_{t_0}^{s} r(t, x(t), \alpha(t))\ dt \tag{3.3}$$

subject to

$$\begin{cases} \dot{x}(t) = f(t, x(t), \alpha(t)), \\ x(t_0) = x_0, \end{cases} \tag{3.4}$$

for $\alpha \in \mathcal{C}([t_0, s], \mathbb{R}^m)$ with fixed endpoints $t_0 < s$, where r and f are continuous in (t, x, α) and have continuous first partial derivatives with respect to x and α for all $(t, x, \alpha) \in [t_0, s] \times \mathbb{R}^n \times \mathbb{R}^m$. Suppose that $\alpha^* \in \mathcal{C}([t_0, s], \mathbb{R}^m)$ is a local minimizer for the problem with respect to the $\|\cdot\|_\infty$ norm,, and let $x^* \in \mathcal{C}^1([t_0, s], \mathbb{R}^n)$ denote the corresponding response. Then, there is a vector function $p^* \in \mathcal{C}^1([t_0, s], \mathbb{R}^n)$ such that the triple (α^*, x^*, p^*) satisfies (3.4)*, the system*

$$\begin{cases} \dot{p}(t) = -\nabla_x r(t, x(t), \alpha(t)) - \nabla_x f(t, x(t), \alpha(t)) \cdot p(t), \\ p(s) = 0; \end{cases} \tag{3.5}$$

and

$$0 = \nabla_\alpha r(t, x(t), \alpha(t)) + \nabla_\alpha f(t, x(t), \alpha(t)) \cdot p(t) \tag{3.6}$$

for $t_0 \leq t \leq s$.

These equations are known collectively as the Euler-Lagrange equations, and (3.5) is often referred to as the adjoint equation (or the costate equation).

PROOF. Consider a one-parameter family of comparison controls

$$v(t;\eta) := \alpha(t) + \eta q(t)$$

where $q(t) \in \mathcal{C}([t_0, s], \mathbb{R}^m)$ is some fixed function, and $\eta \in \mathbb{R}$ is a parameter with $|\eta|$ sufficiently small. Based on the continuity and differentiability properties of f , we know that there exists $\eta_0 \geq 0$ such that the response

$$y(t;\eta) \in \mathcal{C}^1([t_0, s], \mathbb{R}^n)$$

associated to $v(t;\eta)$ through 3.4 exists, is unique, and is differentiable with respect to η, for all $\eta \in B_{\eta_0}(0)$ and for all $t \in [t_0, s]$[1]. Clearly, $\eta = 0$ provides the optimal response $y(t;0) \equiv x^*(t)$, $t_0 \leq t \leq s$. Since the control $v(t;\eta)$ is admissible and its associated response is $y(t;\eta)$, we have, remembering that $f(t, y(t;\eta), v(t;\eta)) - \dot{y}(t;\eta) = 0$,

$$\begin{aligned} P(v(t;\eta)) &= \int_{t_0}^{s} \Big[r(t, y(t;\eta), v(t;\eta)) + p(t) \cdot [f(t, y(t;\eta), v(t;\eta)) - \dot{y}(t;\eta)] \Big]\ dt \\ &= \int_{t_0}^{s} [r(t, y(t;\eta), v(t;\eta)) + p(t) \cdot f(t, y(t;\eta), v(t;\eta)) + \dot{p}(t) \cdot y(t;\eta)]\ dt \\ &\quad - p(s) \cdot y(s;\eta) + p(t_0) \cdot y(t_0;\eta), \end{aligned}$$

[1] See [**4**], Appendix A.5

3.4.2. Extensions of the Pontryagin maximum principle. In this subsection, we shall treat two extensions of the PMP. The first extension is for the case where the terminal condition $x(s) = x_s$ is replaced by the target set condition $x(s) \in X_s \subseteq \mathbb{R}^n$. The second extension is to non-autonomous problems, and makes use of the former. Regarding target set terminal conditions, we have the following theorem:

THEOREM 3.9 (**Transversal Conditions**). *Consider the optimal control problem to minimize the functional*

$$P(\alpha, s) = \int_{t_0}^{s} r(x(t), \alpha(t))\, dt \tag{3.24}$$

subject to

$$\begin{cases} \dot{x}(t) = f(x(t), \alpha(t)) \\ x(t_0) = x_0, \\ x(s) \in X_s \end{cases} \tag{3.25}$$

$$\alpha(t) \in A, \tag{3.26}$$

with fixed initial time t_0 and free terminal time s, and with X_s a smooth manifold of dimension $n_s \leq n$. Let r and f be continuous in (x, α) and have continuous first partial derivatives with respect to x, for all $(x, \alpha) \in \mathbb{R}^n \times \mathbb{R}^m$. Suppose that $(\alpha^*, s^*) \in \mathcal{C}([t_0, T]; \mathbb{R}^m) \times [t_0, T)$ is a minimizer for the problem, and let z^* denote the optimal extended response. Then there exists a piecewise continuously differentiable vector function $\bar{p}^* = ({p_0}^*, {p_1}^*, \ldots, {p_n}^*) \neq (0, 0, \ldots, 0)$ solving 3.19 and satisfying conditions (i) and (ii) of Theorem 3.7. Moreover, $p^*(s^*) = (p_1^*(s^*), p_2^*(s^*), \ldots, p_n^*(s^*))$ is orthogonal to the tangent plane, $T_{x^*(s^*)}X_s$, to X_s at $x^*(s^*)$:

$$p^*(s^*) \cdot d = 0, \qquad \forall d \in T_{x^*(s^*)}X_s$$

Notice that when the set X_s degenerates into a point, the transversality condition at s^* can be replaced by the condition that the optimal response x pass through this point, as in Theorem 3.7.

In order to state the PMP for non-autonomous problems. We shall consider the optimal control problem in the same form as in theorem 3.7, but for the case in which r and f depend explicitly on time (the control region A is assumed independent of time). Thus, the system equations and the cost functional take the form

$$P(\alpha) = \int_{t_0}^{s} r(t, x(t), \alpha(t))\, dt$$

subject to

$$\dot{x}(t) = f(t, x(t), \alpha(t))$$

In order to solve this problem, we shall introduce yet another auxiliary variable, x_{n+1}, defined by

$$\dot{x}_{n+1}(t) = 1; \quad x_{n+1}(t_0) = t_0.$$