# 3. Matrix Algebra

## Unit matrices

The unit matrix $I$ of order $n$ is a square matrix with all diagonal elements equal to one and all off-diagonal elements zero, i.e., $(I)_{ij} = \delta_{ij}$. If $A$ is a square matrix of order $n$, then $AI = IA = A$. Also $I = I^{-1}$.
$I$ is sometimes written as $I_n$ if the order needs to be stated explicitly.

## Products

If $A$ is a $(n \times l)$ matrix and $B$ is a $(l \times m)$ then the product $AB$ is defined by

$$(AB)_{ij} = \sum_{k=1}^{l} A_{ik} B_{kj}$$

In general $AB \neq BA$.

## Transpose matrices

If $A$ is a matrix, then transpose matrix $A^T$ is such that $(A^T)_{ij} = (A)_{ji}$.

## Inverse matrices

If $A$ is a square matrix with non-zero determinant, then its inverse $A^{-1}$ is such that $AA^{-1} = A^{-1}A = I$.

$$(A^{-1})_{ij} = \frac{\text{transpose of cofactor of } A_{ij}}{|A|}$$

where the cofactor of $A_{ij}$ is $(-1)^{i+j}$ times the determinant of the matrix $A$ with the $j$-th row and $i$-th column deleted.

## Determinants

If $A$ is a square matrix then the determinant of $A$, $|A|$ ($\equiv \det A$) is defined by

$$|A| = \sum_{i,j,k,\ldots} \epsilon_{ijk\ldots} A_{1i} A_{2j} A_{3k} \ldots$$

where the number of the suffixes is equal to the order of the matrix.



## 2×2 matrices

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then,

$$|A| = ad - bc \qquad A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

## Product rules

$$(AB \ldots N)^T = N^T \ldots B^T A^T$$

$$(AB \ldots N)^{-1} = N^{-1} \ldots B^{-1} A^{-1} \qquad \text{(if individual inverses exist)}$$

$$|AB \ldots N| = |A|\,|B| \ldots |N| \qquad \text{(if individual matrices are square)}$$

## Orthogonal matrices

An orthogonal matrix $Q$ is a square matrix whose columns $q_i$ form a set of orthonormal vectors. For any orthogonal matrix $Q$,

$$Q^{-1} = Q^T, \quad |Q| = \pm 1, \quad Q^T \text{ is also orthogonal.}$$

# 7. Hyperbolic Functions

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \qquad \text{valid for all } x$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \text{valid for all } x$$

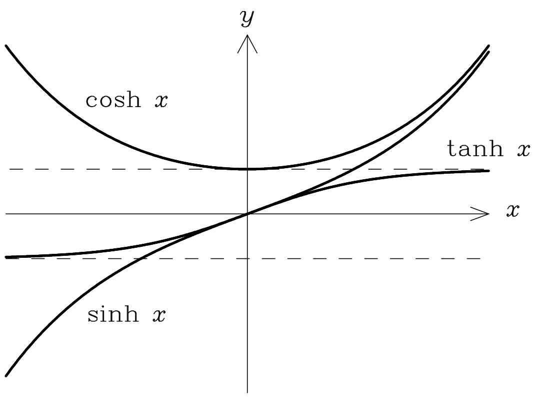
$\cosh ix = \cos x$ | $\cos ix = \cosh x$

$\sinh ix = i \sin x$ | $\sin ix = i \sinh x$

$\tanh x = \dfrac{\sinh x}{\cosh x}$ | $\operatorname{sech} x = \dfrac{1}{\cosh x}$

$\coth x = \dfrac{\cosh x}{\sinh x}$ | $\operatorname{cosech} x = \dfrac{1}{\sinh x}$

$\cosh^2 x - \sinh^2 x = 1$




For large positive $x$:

$$\cosh x \approx \sinh x \to \frac{e^x}{2}$$

$$\tanh x \to 1$$

For large negative $x$:

$$\cosh x \approx -\sinh x \to \frac{e^{-x}}{2}$$

$$\tanh x \to -1$$



## Relations of the functions

$\sinh x = -\sinh(-x)$ | $\operatorname{sech} x = \operatorname{sech}(-x)$

$\cosh x = \cosh(-x)$ | $\operatorname{cosech} x = -\operatorname{cosech}(-x)$

$\tanh x = -\tanh(-x)$ | $\coth x = -\coth(-x)$

$$\sinh x = \frac{2\tanh(x/2)}{1 - \tanh^2(x/2)} = \frac{\tanh x}{\sqrt{1-\tanh^2 x}} \qquad \cosh x = \frac{1+\tanh^2(x/2)}{1-\tanh^2(x/2)} = \frac{1}{\sqrt{1-\tanh^2 x}}$$

$\tanh x = \sqrt{1 - \operatorname{sech}^2 x}$ | $\operatorname{sech} x = \sqrt{1 - \tanh^2 x}$

$\coth x = \sqrt{\operatorname{cosech}^2 x + 1}$ | $\operatorname{cosech} x = \sqrt{\coth^2 x - 1}$

$\sinh(x/2) = \sqrt{\dfrac{\cosh x - 1}{2}}$ | $\cosh(x/2) = \sqrt{\dfrac{\cosh x + 1}{2}}$

$$\tanh(x/2) = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$$

$\sinh(2x) = 2 \sinh x \cosh x$ | $\tanh(2x) = \dfrac{2\tanh x}{1 + \tanh^2 x}$

$$\cosh(2x) = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

$\sinh(3x) = 3 \sinh x + 4 \sinh^3 x$ | $\cosh 3x = 4\cosh^3 x - 3\cosh x$

$$\tanh(3x) = \frac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$$

# 13. Functions of Several Variables

If $\phi = f(x, y, z, \ldots)$ then $\dfrac{\partial\phi}{\partial x}$ implies differentiation with respect to $x$ keeping $y, z, \ldots$ constant.

$$\mathrm{d}\phi = \frac{\partial\phi}{\partial x}\mathrm{d}x + \frac{\partial\phi}{\partial y}\mathrm{d}y + \frac{\partial\phi}{\partial z}\mathrm{d}z + \cdots \quad \text{and} \quad \delta\phi \approx \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z + \cdots$$

where $x, y, z, \ldots$ are independent variables. $\dfrac{\partial\phi}{\partial x}$ is also written as $\left(\dfrac{\partial\phi}{\partial x}\right)_{y,\ldots}$ or $\left.\dfrac{\partial\phi}{\partial x}\right|_{y,\ldots}$ when the variables kept constant need to be stated explicitly.

If $\phi$ is a well-behaved function then $\dfrac{\partial^2\phi}{\partial x\,\partial y} = \dfrac{\partial^2\phi}{\partial y\,\partial x}$ etc.

If $\phi = f(x, y)$,

$$\left(\frac{\partial\phi}{\partial x}\right)_y = \frac{1}{\left(\dfrac{\partial x}{\partial\phi}\right)_y}, \qquad \left(\frac{\partial\phi}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_\phi \left(\frac{\partial y}{\partial\phi}\right)_x = -1.$$

## Taylor series for two variables

If $\phi(x, y)$ is well-behaved in the vicinity of $x = a, y = b$ then it has a Taylor series

$$\phi(x, y) = \phi(a + u, b + v) = \phi(a, b) + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + \frac{1}{2!}\left(u^2\frac{\partial^2\phi}{\partial x^2} + 2uv\frac{\partial^2\phi}{\partial x\,\partial y} + v^2\frac{\partial^2\phi}{\partial y^2}\right) + \cdots$$

where $x = a + u, y = b + v$ and the differential coefficients are evaluated at $x = a, \quad y = b$

## Stationary points

A function $\phi = f(x, y)$ has a stationary point when $\dfrac{\partial\phi}{\partial x} = \dfrac{\partial\phi}{\partial y} = 0$. Unless $\dfrac{\partial^2\phi}{\partial x^2} = \dfrac{\partial^2\phi}{\partial y^2} = \dfrac{\partial^2\phi}{\partial x\,\partial y} = 0$, the following conditions determine whether it is a minimum, a maximum or a saddle point.

$$\left.\begin{array}{ll} \text{Minimum:} & \dfrac{\partial^2\phi}{\partial x^2} > 0, \text{ or } \dfrac{\partial^2\phi}{\partial y^2} > 0, \\ \text{Maximum:} & \dfrac{\partial^2\phi}{\partial x^2} < 0, \text{ or } \dfrac{\partial^2\phi}{\partial y^2} < 0, \end{array}\right\} \text{ and } \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} > \left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2$$

$$\text{Saddle point:} \quad \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} < \left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2$$

If $\dfrac{\partial^2\phi}{\partial x^2} = \dfrac{\partial^2\phi}{\partial y^2} = \dfrac{\partial^2\phi}{\partial x\,\partial y} = 0$ the character of the turning point is determined by the next higher derivative.



## Changing variables: the chain rule

If $\phi = f(x, y, \ldots)$ and the variables $x, y, \ldots$ are functions of independent variables $u, v, \ldots$ then

$$\frac{\partial\phi}{\partial u} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial u} + \cdots$$
$$\frac{\partial\phi}{\partial v} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial v} + \cdots$$

etc.