Function of bounded variation

Def of bounded variation.

The function $\alpha : [a,b] \to \mathbb{R}$ is said to be bounded variation iff there is a constant $M > 0$ s.t

$$\sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| \le M$$

for all partition $P = \{x_0, x_1, \cdots x_n\}$ of $[a,b]$

Example If $\alpha : [a,b] \to \mathbb{R}$ is monotonically increasing then for any partition $P = \{x_0, x_1, \cdots x_n\}$ of $[a,b]$

$$\sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| = \sum_{i=1}^{n} \{\alpha(x_i) - \alpha(x_{i-1})\}$$

$$= \alpha(x_n) - \alpha(x_0)$$

$$= \alpha(b) - \alpha(a)$$

Thus α is of bounded variation & $V_f(a,b) = \alpha(b) - \alpha(a)$

Example If $\alpha : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) with $\sup_{a<x<b} |\alpha'(x)| \le M$ then for any partition $P = \{x_0, x_1, \cdots x_n\}$ of $[a,b]$ we have by mean value theorem.

$$= \sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})|$$

$$= \sum_{i=1}^{n} |\alpha'(t_i)[x_i - x_{i-1}]| \le \sum_{i=1}^{n} M[x_i - x_{i-1}] = M(b-a)$$

Thus α is of bounded variation and $V_f(a,b) \le M(b-a)$

Def Total bounded variation

If $\alpha : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$ then the total variation of α on $[a,b]$ is define to be $V_\alpha(a,b) = \sup \{ \sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| \ P = \{x_0, x_1, \cdots x_n\}$ is a partition of $[a,b]\}$.