Theorem:-Every bounded monotonic sequence $\{a_n\}_{n=1}^{\infty}$ is convergent.

Proof:-Assume that $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence.

Since $\{a_n\}_{n=1}^{\infty}$ is bound, the set $S = \{a_1, a_2, \ldots \ldots, a_n, \ldots \ldots \ldots\}$ has an upper bound.

By completeness axiom it has a least upper bound L.

Let $\varepsilon > 0$,then $L - \epsilon$ is not an upper bound of S since L is the least upper bound.

Thus $\exists$ a number a_N in S such that $L - \epsilon < a_N \leq L$ for some integer N.

Since the given sequence is increasing, we have $a_N \leq a_n$ for $n \geq N$.

Hence $L - \varepsilon < a_N \leq a_n \leq L; \quad \forall\, n \geq N.$

Thus $-\varepsilon < a_N - L \leq a_n - L \leq L - L; \quad \forall\, n \geq N$

$\Rightarrow -\varepsilon < a_n - L \leq 0; \quad \forall\, n \geq N$

$\Rightarrow -\varepsilon < a_n - L < \varepsilon; \quad \forall\, n \geq N$

$\Rightarrow |a_n - L| < \varepsilon; \quad \forall\, n \geq N$

$\therefore \lim_{n\to\infty} a_n = L.$

Hence $\{a_n\}_{n=1}^{\infty}$ converges to L.

Example:-7) Let $\{a_n\}_{n=0}^{\infty}$ be the sequence defined by $a_0 = 0$ *and* $a_{n+1} = a_n^2 + \frac{1}{4}, \quad \forall\; n \geq 0.$

Show that $\{a_n\}_{n=0}^{\infty}$ converges.

Solution:-Using the above theorem it is enough to prove that the sequence is bounded and increasing.

(i) To prove boundedness

Claim that $|a_n| < \frac{1}{2}; \;\; n \geq 0$

$a_0 = 0 \;\Rightarrow |a_0| = 0 < \frac{1}{2}$

Assume that $|a_n| < \frac{1}{2}; \;\; \forall\, n \geq 0.$

Then $|a_{n+1}| = a_n^2 + \frac{1}{4} < (\frac{1}{2})^2 + \frac{1}{4} = \frac{1}{2} \qquad [\because a_n^2 = |a_n|^2 < (\frac{1}{2})^2]$

On the other hand if the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ diverges, then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges. In this case $\lim_{n\to\infty} s_n = \pm\infty$ or $\lim_{n\to\infty} s_n$ is not unique ,

then $\sum_{i=1}^{\infty} a_i$ *is diverges*.

Note:-If $\lim_{n\to\infty} s_n$ is not unique then $\sum a_n$ **oscillates**.

Example:-1) Determine whether the series $\sum_{n=1}^{\infty} n$ is convergent or divergent.

If it converges determine its value .

Solution:-The sequence of partial sum is $s_n = \sum_{i=1}^{n} i$

This is known series and its value can be shown to be

$$s_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

So, to determine if the series is convergent we will first need to see if the sequence of partial sums,

$$\left\{\frac{n(n+1)}{2}\right\}_{n=1}^{\infty} \text{ is convergent or divergent.}$$

Now, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$.

Therefore, the sequence of partial sums diverges to ∞. So the series *diverges*.

Example:-2) Show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots$converges and find the sum.

Solution:-By partial fraction $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ $for\ n \geq 1$.

Hence $s_n = \sum_{i=1}^{n} \frac{1}{n(n+1)} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \cdots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

$$\therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$$

Hence $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

Note:-The series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ is called a telescoping series because when we write the partial sums all but the first and the last terms cancel.

Example:-3) Determine whether the series $\sum_{n=1}^{\infty}\frac{1}{2^n}$is convergent or divergent. If it converges determine its value .

Solution: - The sequence of partial sum

$$s_n = \sum_{i=1}^{n}\frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

So, to determine if the series is convergent we will first need to see if the sequence of partial sums,

$$\{\, 1 - \frac{1}{2^n}\}_{n=1}^{\infty} \text{ is convergent or divergent.}$$

In this case the limit of the sequence of partial sums is,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1$$

Therefore, the sequence of partial sums is convergent and the series will also be convergent. The value of the series is,

$$\sum_{n=1}^{\infty}\frac{1}{2^n} = 1$$

Example:-4) consider the series $1 + 2 - 3 + 1 + 2 - 3 + \cdots$

Here $\lim_{n\to\infty} s_n = 1 \; or \; 3 \; or \; 0.$

Hence $\lim_{n\to\infty} s_n$ is not unique. The series oscillates.

Activities

Determine whether the following series is convergent or divergent. If it converges determine its value .

a) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1}$

b) $\sum_{n=1}^{\infty}(-1)^n$

c) $\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$

d) $\sum\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$

Note:-The converse of the above theorem is not true in general.

i.e If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ may or may not converge.

Example: -7) The series $\sum_{n=1}^{\infty}\frac{1}{n}$ does not converge but $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0.$

Theorem :-(The Test for Divergence)

If $\lim_{n\to\infty} a_n \neq 0$ (does not exist) then the series $\sum_{n=1}^{\infty} a_n$ is diverges.

Proof:-Suppose $\sum_{n=1}^{\infty} a_n$ is diverges, then $\lim_{n\to\infty} a_n = 0$ by the above theorem

Hence by contra positive $\sum_{n=1}^{\infty} a_n$ is diverges

Example:- 8) Show that the series $\sum_{n=1}^{\infty}\frac{n^2}{5n^2+4}$ diverges.

Solution:- $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^2}{5n^2+4} = \lim_{n\to\infty}\frac{1}{5+4/n^2} = \frac{1}{5} \neq 0$

So the series $\sum_{n=1}^{\infty}\frac{n^2}{5n^2+4}$ diverges by the Divergence Test theorem.

Activities

Show that whether the following series are convergent or divergent.

a) $\sum_{n=1}^{\infty}\frac{n^2-2}{n^2+5n}$ b) $\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$

Some types of infinite series

1. **Harmonic series**

Definition:-the series $\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ =1 is called a harmonic series.

Note:- the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent (**see example 7 above**)

2. **The Geometric Series**

Definition:-The series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots + ar^{n-1}$ ... is called a geometric series. r is called the ratio of the geometric series .

Theorem:- the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ where $a \neq 0$ converges if $|r| < 1$ and its sum is

Therefore, $\lim_{n\to\infty} s_n$ does not exist since it oscillates between 0 and a.

Hence geometric series diverges.

Case (iii) If $|r| > 1$,then $r^n \to \infty \; as \; n \to \infty$,

Then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(r^n-1)}{r-1} = \frac{a}{1-r}\lim_{n\to\infty} r^n - \frac{a}{1-r} = \infty$

The series is diverges.

Therefore, for $|r| \geq 1$, the geometric series is divergent.

Hence theorem proved.

Example:- 9) Determine whether the each following series converges or diverges. In case of convergence find the sum .

a) $\sum_{n=1}^{\infty} \frac{1}{4^{n-1}}$ b) $\sum_{n=1}^{\infty} 3\left(\frac{-2}{3}\right)^{n-1}$ c) $\sum_{n=1}^{\infty} \frac{4.5^{n-1}}{4^{n-1}}$

Solution:-a) $\sum_{n=1}^{\infty} \frac{1}{4^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n-1}$ which is a geometric series with $r = \frac{1}{4}$ and $a = 1$.

Since $|r| = \left|\frac{1}{4}\right| < 1$, the series converges

$\therefore \sum_{n=1}^{\infty} \frac{1}{4^{n-1}} = \frac{a}{1-r} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$

b) $\sum_{n=1}^{\infty} 3\left(\frac{-2}{3}\right)^{n-1}$ is a geometric series with $r = \frac{-2}{3}$ and $a = 3$.

Since $|r| = \left|\frac{-2}{3}\right| < 1$, the series converges

$\therefore \sum_{n=1}^{\infty} 3\left(\frac{-2}{3}\right)^{n-1} = \frac{a}{1-r} = \frac{3}{1-\left(\frac{-2}{3}\right)} = \frac{9}{5}$

c) $\sum_{n=1}^{\infty} \frac{4.5^{n-1}}{4^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{5}{4}\right)^{n-1}$ Which is a geometric series with $r = \frac{5}{4}$ and $a = 4$.

Since $|r| = \left|\frac{5}{4}\right| > 1$, the series diverges.

The above theorem can be used to express as a non terminating repeating decimals as a rational number.

Example:-10) Express decimals as a rational number

Activity

Find radius and interval of convergence of the following series

$a)\sum_{n=0}^{\infty} x^n$ $\qquad b)\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\qquad c)\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{2}{3}\right)^n \frac{x^n}{n}$

3.3 Algebraic operations on convergent power series

We may also add, subtract, multiply and divide the series just like polynomials. The following theorem states the rules for algebraic operations on convergent power series.

Theorem:-The Algebra of Power Series

Assume that $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \quad and$

$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots \qquad for\ |x| < R.\ \text{Then}$$

$for\ |x| < R$, we have

(i) $f(x) + g(x) = \sum_{n=0}^{\infty}(a_n + b_n)x^n$

(ii) $f(x) - g(x) = \sum_{n=0}^{\infty}(a_n - b_n)x^n$

(iii) $f(x)\cdot g(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots$

(iv) $f(x)/g(x)$ is obtained by long division, if $g(x) \neq 0\ for\ |x| < R$

3.4 power series Representation of functions

In this section we will see how to represent certain types of functions as a sum of a power series by manipulating geometric series or by differentiating or integrating such a series. Representation of functions as a power series is useful for integrating functions that do not have elementary anti-derivatives, for solving differential equations and for approximating functions by polynomials.

Consider the equation $1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n$ with $|x| < 1$ -------------- (i)

Observe that the series is a geometric series with first term $a = 1$ and common ratio $r = x$.

Since $|r| = |x| < 1$, it converges and equation (i) gives

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{with } |x| < 1 \qquad \text{---------------- (ii)}$$

Hence, equation (ii) shows the function $f(x) = \frac{1}{1-x}$ as a sum of power series.

Example:-1) Express $\frac{1}{1+x^2}$ as the sum of a power series and find the interval of convergence.

Solution:-We know that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ ---------------- (i)

Now replacing x by $-x^2$, we get

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$= 1 - x^2 + x^4 - x^6 + \cdots$$

Because this is a geometric series it converges when $|-x^2| < 1$, that is $x^2 < 1$ or $|x| < 1$.

Therefore, the interval of convergence is $(-1,1)$.

Example:-2) Express $\frac{1}{x-5}$ as the sum of a power series and find the interval of convergence.

Solution:-Given $f(x) = \frac{1}{x-5}$

We know that $\frac{1}{x-5} = \frac{1}{-(5-x)} = \frac{1}{-5(1-\frac{x}{5})} = \frac{-1}{5}\left(\frac{1}{1-\frac{x}{5}}\right)$

$$f(x) = \frac{1}{x-5} = \frac{-1}{5}\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n = -\sum_{n=0}^{\infty}\frac{1}{5}\cdot\frac{x^n}{5^n}$$

$$= -\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}$$

This series converges for $\left|\frac{x}{5}\right| < 1$

Therefore, the power series representation of $f(x) = \frac{1}{x-5}$ is $-\sum_{n=0}^{\infty}\frac{x^n}{5^{n+1}}$ and the interval of convergence is $(-5,5)$.

Example:-Express $\frac{x}{4x+1}$ as the sum of a power series and find the interval of convergence.

Solution:-Given $f(x) = \frac{x}{4x+1}$

We know that $\frac{x}{4x+1} = x\cdot\frac{1}{1-(-4x)} = x\sum_{n=0}^{\infty}(-4x)^n = \sum_{n=0}^{\infty}(-4)^n x^{n+1}$,

The series converges for $|-4x| < 1 \Rightarrow 4|x| < 1 \Rightarrow |x| < \frac{1}{4}$

Therefore, the power series representation of $f(x) = \frac{x}{4x+1}$ is $\sum_{n=0}^{\infty}(-4)^n x^{n+1}$, and the interval of convergence is $(\frac{-1}{4},\frac{1}{4})$

Activity

Find a power series representation for the following functions and determine its interval of convergence

a) $f(x) = \frac{1}{1+x^3}$ b) $g(x) = \frac{x}{5-x}$ c) $f(x) = \frac{1}{1+x^3}$

3.5 Differentiation and integration of power series

A power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ with a nonzero radius of convergence is always differentiable and the derivative is obtained from $\sum_{n=0}^{\infty} c_n (x-a)^n$ by differentiating term by term, the way we differentiate polynomials.

Theorem:-If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$, then the

function f defined by $f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n (x-a)^n$

Is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(a) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$

(b) $\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$

$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the power series in Equations (a) and (b) are both R.

Note:-Equations (a) and (b) can be rewritten in the form

- $\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n (x-a)^n]$
- $\int\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right]dx = \sum_{n=0}^{\infty} \int c_n (x-a)^n dx$

Example:-1) Express $\frac{1}{(1-x)^2}$ as a power series by differentiating the equation

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n$ and find radius of convergence.

Solution:-Differentiating both sides of the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n$$

We get $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n$

Properties

1. If T is a period of f, then nT of T is also a period of f for any positive integer n, that is, $f(x+nT) = f(x)$ for all real number x in the domain of f.
2. If the period of $f(x)$ is T, then the period of $f(ax+b)$ is $\frac{T}{a}$.
3. If $f(x)$ and $g(x)$ are periodic functions having the same period T, Then the period $h(x) = c_1 f(x) + c_2 g(x)$ is also T.
4. If $f(x) = c$, where c is a constant, then f is periodic and every $T > 0$ is a period. Thus there exists no smallest period, so does not have the fundamental period.
5. The sum of the number of trigonometric functions in $\sin nx$ and $\cos nx$ is also a periodic function with period the least common multiple of the periods of each term of the sum.

Example:-3) $\boldsymbol{f(x)} = sinx + \frac{1}{2}sin2x + \frac{1}{3}sin3x + \frac{1}{4}sin4x$ is a periodic function.

The periods of the terms are $2\pi, \pi, \frac{2\pi}{3}$ and $\frac{\pi}{2}$ respectively. So the period of the given function is the least common multiple of these periods which is equal to 2π

Example:-4) $f(x) = \cos(\pi x) + \frac{1}{2}\cos(2\pi x)$ is a periodic function with period 2.

Trigonometric series

Definition: - the series the form $\frac{a_0}{2} + a_1 cosx + b_1 sinx + a_2 cos2x + b_2 sin2x + \cdots$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k coskx + b_k sinkx) \text{ is called a trigonometric series.}$$

A trigonometric series is a periodic function with period 2π. the convergence and divergence of the series depends on the value of x chosen and the coefficients a_k and b_k.

Trigonometric values

Let k be an integer then,

1. $\sin k\pi = 0 = \sin(k \pm 1)\pi$
2. $\cos k\pi = (-1)^k$
3. $\sin\left(k + \frac{1}{2}\right)\pi = (-1)^k$
4. $\cos\left(k + \frac{1}{2}\right)\pi = 0$
5. $\cos 2(k \pm 1)\pi = 1$
6. $\cos(k \pm 1)\pi = (-1)^{k \pm 1}$

Note: - The Half-Range cosine series in $[0,\pi]$ is obtained by putting $l=\pi$.

Half-Range sine series

Let $f(x)$ be defined in the range $[0,l]$. in order to obtain Fourier series we construct a new function in such a way that the function $f(x)$ plus its extension yield an odd function.

Define $$F(x)=f(x) \quad for \quad 0<x<l$$

$$=-f(-x) \quad for \ -l<x<0$$

So that $F(x)$ is an odd function in $[-l,l]$. Hence the Fourier series of $F(x)$ contains only sine terms.

$$\therefore \qquad F(x)=\sum_{k=1}^{\infty} b_k \sin\frac{k\pi x}{l}$$

Where $$a_0=\frac{1}{l}\int_{-l}^{l} F(x)dx = 0, \quad b_k=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{k\pi x}{l}dx=0$$

$$b_k=\frac{1}{l}\int_{-l}^{l} F(x)\sin\frac{k\pi x}{l}dx=\frac{2}{l}\int_0^l F(x)\sin\frac{k\pi x}{l}dx=\frac{2}{l}\int_0^l f(x)\sin\frac{k\pi x}{l}dx$$

Since $F(x)=f(x)$ in $[0,l]$

In this case the Fourier series is called **Half-Range Fourier Sine series** which is defined

as $$f(x)=\sum_{k=1}^{\infty} b_k \sin\frac{k\pi x}{l}$$

Where $$a_k=\frac{2}{l}\int_0^l f(x)\sin\frac{k\pi x}{l}dx$$

Note: - The Half-Range sine series in $[0,\pi]$ is obtained by putting $l=\pi$.

Example:-6) Find the Half-Range cosine and Sine series for the function

$$f(x)=\begin{cases} x & for \quad 0<x<1 \\ 2-x & for \ 1<x<2 \end{cases}$$

Solution: - Here $l=2-0=2$

i) the Half-Range cosine series

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos\frac{k\pi x}{l}$$

Example:-Find the domain and range and sketch the graph of $g(x,y) = \sqrt{9 - x^2 - y^2}$

Solution:-The domain of g is

$D = \{(x,y)|9 - x^2 - y^2 \geq 0\} = \{(x,y)|x^2 + y^2 \leq 9\}$

Which is the disk with center $(0,0)$ and radius 3.

The range of g is $\{z|z = \sqrt{9 - x^2 - y^2}, (x,y) \in D\}$

Since z is positive square root, $z \geq 0$. also

$9 - x^2 - y^2 \leq 9 \quad \Rightarrow \sqrt{9 - x^2 - y^2} \leq 3$

So the range is$\{z|0 \leq z \leq 3\} = [0,3]$.

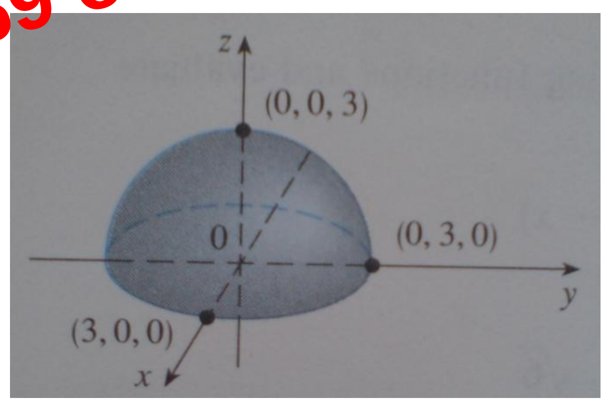
The graph has the equation $z = \sqrt{9 - x^2 - y^2}$,

We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2, or\ x^2 + y^2 + z^2 = 9$,

Which we recognize as an equation of the sphere with center the origin and radius 3. But, since $z \geq 0$, the graph of g is just the top half of this sphere.

Level Curves

So far we have two methods for visualizing functions: arrow diagrams and graphs. A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form *contour curves, or level curves*.

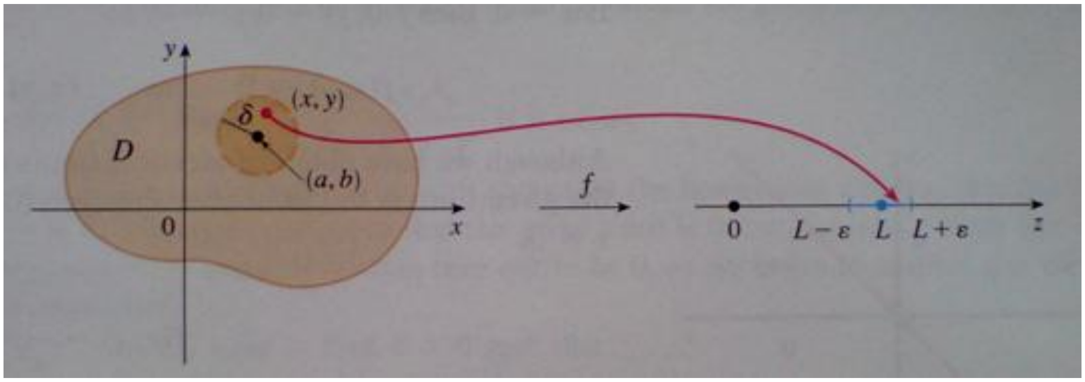


The **level curves** of a function f of two variables are the curves with equations $f(x,y) = k$, where k is a constant (in the range of f)

Note

- A level curve $f(x,y) = k$ is the locus of all points at which f takes on a given value k. In other words, it shows where the graph of f has height k.

an arrow diagram. If any small interval $(L-\varepsilon, L+\varepsilon)$ is given around L, then we can find a disk D_δ with center (a,b) and radius $\delta > 0$ such that f maps all the points in D_δ [$except\ possibly\ (a,b)$] into the interval $(L-\varepsilon, L+\varepsilon)$.



Note

The above limit definition refers only to the distance between (x,y) and (a,b). It does not refer to the direction of approach. Therefore, if the limit exists, then $f(x,y)$ must approach the same limit no matter how (x,y) approaches (a,b). Thus, if we can find two different paths of approach along which $f(x,y)$ has different limits, then it follows that $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

If $f(x,y) \to L_1 as\ (x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2 as\ (x,y) \to (a,b)$ along a path C_2, $where\ L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example:-Find $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ if it exists.

Solution:-Let $f(x,y) = x^2 - y^2/x^2 + y^2$.

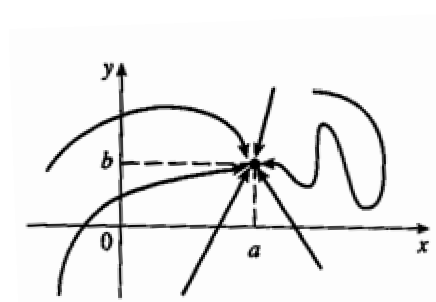
First let us approach (0, 0) along the $x-axis$. Then $y=0$ gives $f(x,0) = x^2/x^2 = 1\ \forall x \neq 0$, so

$$f(x,y) \to 1 \qquad as \qquad (x,y) \to (0,0)\ along\ the\ x-axis$$

We now approach along the $y-axis$ by putting $x=0$. Then $f(x,0) = -y^2/y^2 = -1\ \forall y \neq 0$, so

$$f(x,y) \to -1 \qquad as \qquad (x,y) \to (0,0)\ along\ the\ y-axis$$

Since f has two different limits along two different lines, the given limit doesn't exist.



$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} $\left(or\ \frac{\partial^2 f}{\partial y \partial x}\right)$ means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

Example: - Find the second partial derivatives of $f(x, y) = x^3 + x^2y^3 - 2y^2$

Solution: We know that

$$f_x(x,y) = 3x^2 + 2xy^3 \qquad\qquad f_y(x,y) = 3x^2y^2 - 4y$$

Therefore

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3) = 6x + 2y^3$$

$$f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3) = 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y) = 6xy^2$$

$$f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y) = 6x^2y - 4$$

Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} *and* f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$

5.5 Tangent planes and Total differential

Total Differential

The **differentials** dx and dy are independent variables; that is, they can be given any values. Then the **differential** dz, also called the **total differential**, is defined by

$$dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial z}dx + \frac{\partial z}{\partial y}dy$$

5.10 Extreme values under constraint conditions: Lagrange's multiplier

Let f and g differentiable at (x_0,y_0). Let c be the curve $g(x,y) = c$ that contains (x_0,y_0). Assume that c in smooth and that (x_0,y_0) is not an end point of the curve. If grad $g(x_0,y_0) \neq 0$ and f has an extreme value on c at (x_0,y_0) then there is a number λ such that $grad$ $f(x_0,y_0) = \lambda\, grad g(x_0,y_0)$. The number λ is called Lagranges Multiplier.

Example:- Let $f(x,y) = x^2 + 4y^3$. Find the extreme values of f on the ellipse $x^2 + 2y^2 = 1$ and points at which they occur.

Solution:- Let $g(x,y) = x^2 + 2y^2$ so that the constraint is $g(x,y) = x^2 + 2y^2 = 1$

Since grad $f(x,y) = 2xi + 12y^2 j$

Grad $g(x,y) = 2xi + 4yj$

Use these equations to find x, y constraint $x^2 + 2y^2 = 1$ ------------ (1)

$$gradf(x,y) = \lambda grad g(x,y)$$

$$2xi + 12y^2 j = \lambda(2xi + 4yj)$$

$$2x = \lambda(2x) \text{ ------------ (2)}$$

$$12y^2 = 4y\lambda \text{ ------------ (3)}$$

By equation (2), either $x = 0$ or $\lambda = 1$

If $x = 0$ from (1) $y = \pm\frac{1}{\sqrt{2}} \Rightarrow \left(0,\frac{1}{\sqrt{2}}\right), (0,-\frac{1}{\sqrt{2}})$

If $\lambda = 1$ from (3) $12y^2 = 4y \Rightarrow y = 0$ or $y = \frac{1}{3}$

If $y = 0 \Rightarrow$ from (1) $x = \pm 1 \Rightarrow (1,0), (-1,0)$

If $y = \frac{1}{3} \Rightarrow$ from (1) $x = \pm\frac{\sqrt{7}}{3} \Rightarrow \left(\frac{\sqrt{7}}{3},\frac{1}{3}\right), (-\frac{\sqrt{7}}{3},\frac{1}{3})$

$\therefore$ The extreme values of f occurs at

$$\left(0,\frac{1}{\sqrt{2}}\right)\left(0,-\frac{1}{\sqrt{2}}\right)(1,0)(-1,0)\left(\frac{\sqrt{7}}{3},\frac{1}{3}\right)\left(-\frac{\sqrt{7}}{3},\frac{1}{3}\right)$$

Now, $f\left(0,\frac{1}{\sqrt{2}}\right) = \sqrt{2}, \quad f\left(0,-\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$

$$f(1,0) = 1 = f(-1,0); \qquad f\left(\frac{\sqrt{7}}{3},\frac{1}{3}\right) = \frac{25}{27} = f\left(-\frac{\sqrt{7}}{3},\frac{1}{3}\right)$$

We will compute the double integral by first computing $\int_c^d f(x,y)dy$ and we compute this by holding *x* constant and integrating with respect to *y* as if this were a single integral. This will give a function involving only x's which we can in turn integrate.

We've done a similar process with partial derivatives. To take the derivative of a function with respect to *y* we treated the x's as constants and differentiated with respect to *y* as if it was a function of a single variable.

Double integrals work in the same manner. We think of all the x's as constants and integrate with respect to y or we think of all y's as constants and integrate with respect to x.

Let's take a look at some examples.

6.2 Double integrals over rectangular regions

Example1. *Compute each of the following double integrals over the indicated rectangles*

(a) $\iint_R 6xy^2 dA, \qquad R = [2,4]\times[1,2]$

(b) $\iint_R \{x^2y^2 + \cos(\pi x) + \sin(\pi y)\}dA, \qquad R = [-2,1]\times[0,1]$

(c) $\iint_R \frac{1}{(2x+3y)^2}dA, \qquad R = [0,1]\times[1,2]$

(d) $\iint_R [2x - 4y^3]dA, \qquad R = [-5,4]\times[0,3]$

Solutions: a) $\iint_R \mathbf{6xy^2dA} \quad \mathbf{R = [2,4]\times[1,2]}$

It doesn't matter which variable we integrate with respect to first, we will get the same answer regardless of the order of integration. To prove that let's work this one with each order to make sure that we do get the same answer.

Method 1:-In this case we will integrate with respect to *y* first. So, the iterated integral that we need to compute is, $\iint_R 6xy^2 dA = \int_2^4\int_1^2 6xy^2 dy\,dx$

When setting these up make sure the limits match up to the differentials. Since the dy is the inner differential (*i.e.* we are integrating with respect to *y* first) the inner integral needs to have *y* limits. To compute this we will do the inner integral first and we typically keep the outer integral around as follows,

$$\begin{aligned}\iint_R 6xy^2 dA &= \int_2^4 (2xy^3)\Big|_1^2 dx\\ &= \int_2^4 (16x - 2x)dx\\ &= \int_2^4 14x\,dx\end{aligned}$$

Solutions:- a. $\iint_D e^{x/y}dA,\quad D = \{(x,y)|1 \le y \le 2,\ y \le x \le y^3\}$

Okay, this first one is set up to just use the formula above so let's do that.

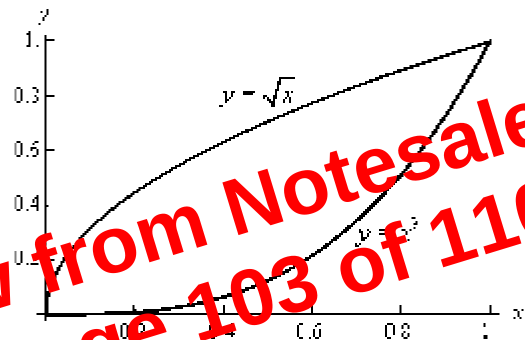
$$\iint_D e^{x/y}dA = \int_1^2 \int_y^{y^3} e^{x/y}\,dx\,dy = \int_1^2 ye^{x/y}\Big|_y^{y^3}\,dy$$

$$= \int_1^2 [ye^{y^2} - ye]dy$$

$$= \left(\frac{1}{2}e^{y^2} - \frac{1}{2}ey^2\right)\Big|_1^2 = \frac{1}{2}e^4 - 2e$$

b. $\iint_D (4xy - y^3)dA$, **D is the region bounded by** $y = \sqrt{x}$ **and** $y = x^3$

In this case we need to determine the two inequalities for *x* and *y* that we need to do the integral. The best way to do this is the graph the two curves. Here is a sketch.



So, from the sketch we can see that the two inequalities are

$$0 \le x \le 1 \qquad x^3 \le y \le \sqrt{x}$$

We can now do the integral

$$\iint_D (4xy - y^3)dA = \int_0^1 \int_{x^3}^{\sqrt{x}} (4xy - y^3)\,dy\,dx$$

$$= \int_0^1 (2xy^2 - \frac{1}{4}y^4)\Big|_{x^3}^{\sqrt{x}}\,dx$$

$$= \int_0^1 (\frac{7}{4}x^2 - 2x^7 + \frac{1}{4}x^{12})\,dx$$

$$= (\frac{7}{12}x^3 - \frac{1}{4}x^8 + \frac{1}{52}x^{13})\Big|_0^1 = \frac{55}{156}$$

c. $\displaystyle\iint_D (6x^2 - 40y)dA$, **D is the triangle with vertices** $(0,3), (1,1),$ **and** $(5,3)$.

We got even less information about the region this time. Let's us start this off by sketching the triangle.

Solution 1:- $\iint_D (6x^2-40y)dA = \iint_{D_1}(6x^2-40y)dA + \iint_{D_2}(6x^2-40y)dA$

$$= \int_0^1 \int_{-2x+3}^{3}(6x^2-40y)dydx + \int_1^5 \int_{\frac{1}{2}x+\frac{1}{2}}^{3}(6x^2-40y)dydx$$

$$= \int_0^1 (6x^2y-20y^2)\Big|_{-2x+3}^{3} dx + \int_1^5 (6x^2y-20y^2)\Big|_{\frac{1}{2}x+\frac{1}{2}}^{3} dx$$

$$= \int_0^1 [12x^3-180+20(3-2x)^2]dx + \int_1^5 [-3x^3+15x^2-180+20(\frac{1}{2}x+\frac{1}{2})^2]dx$$

$$= \left[3x^4-180x-\frac{10}{3}(3-2x)^3\right]\Big|_0^1 + \left[\frac{-3}{4}x^4+5x^3-180x+\frac{40}{3}\left(\frac{1}{2}x+\frac{1}{2}\right)^3\right]\Big|_1^5$$

$$= -\frac{935}{3}$$

Solution 2: This solution will be a lot less work since we are only going to do a single integral.

$$\iint_D (6x^2-40y)dA = \int_1^3 \int_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1}(6x^2-40y)dx\,dy$$

$$= \int_1^3 (2x^3-40xy)\Big|_{-\frac{1}{2}y+\frac{3}{2}}^{2y-1} dy$$

$$= \int_1^3 [100y-100y^2+2(2y-1)^3-2(-\frac{1}{2}y+\frac{3}{2})^3]dy$$

$$= [50y^2-\frac{100}{3}y^3+\frac{1}{4}(2y-1)^4+\left(-\frac{1}{2}y+\frac{3}{2}\right)^4]\Big|_1^3$$

$$= -\frac{935}{3}$$

Exercise

a. Evaluate $\iint\limits_D xy\,dA,\quad D=\{(x,y)|0\le x\le 1, x^2\le y\le \sqrt{x}\}$

b. Evaluate $\iint\limits_D (x+2y)dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1+x^2$

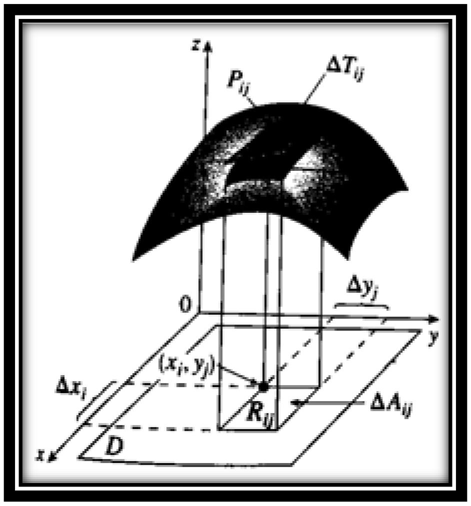
c. Evaluate $\iint\limits_D x\cos y\,dA$, where D is bounded by $y=0, y=x^2,$ and $x=1$

Answers

a. $\frac{1}{12}$ b. $\frac{32}{15}$ c. $\frac{1-\cos 1}{2}$

6.5 Surface Areas

In this section we apply double integral to the problem of computing the area of a surface area. We found the area of a very special type of surface – a surface of revolution – by the method of single-variable calculus. Here we compute the area of a surface with equation z = f(x, y), the graph of a function of two variables.



Let S be a surface with equation z = f(x, y), where f has continuous partial derivatives. For simplicity in deriving the surface area formula, we assume that f(x, y) ≥ 0 and the domain D of f is a rectangle. We consider a partition P of D into small rectangles R_{ij} with areas $\Delta A_{ij} = \Delta x_i \Delta y_j$. If (x_i, y_j) is the corner of R_{ij} closet to the origin, let $P_{ij}[x_i, y_j, f(x_i,\ y_j)]$ be the point on S directly above it. The tangent plane to S at P_{ij} is an approximation to S near P_{ij}. So the area ΔT_{ij} of the part of this tangent plane (a parallelogram) that lies directly above R_{ij} is an approximation to the area ΔS_{ij} of the part of S that lies directly above R_{ij}. Thus the sum $\sum\sum \Delta T_{ij}$ is an approximation to the total area of S, and this approximation appears to improve as $\|P\| \to 0$. Therefore, we define the **surface area** of S to be

(1)
$$\boxed{A(S) = \lim_{\|P\| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}}$$

To find a formula that is more convenient than Equation 1 for computational purpose, we let **a** and **b** be the vectors that start at P_{ij} and lie along the sides of the parallelogram with area ΔT_{ij} (see in following figure). Then $\Delta T_{ij} = |a \times b|$ and $f_x(x_i, y_j)$ and $f_y(x_i, y_j)$ are the slopes of the tangent lines through P_{ij} in the direction of **a** and **b**. Therefore